Thm: A subgp of a free gp is free.

Pf: G free, $H \leq G$.

$G = \pi_1(X)$, $X = \bigvee_\alpha S^1$

So H corresp. to a cover of X, i.e.

$H \simeq \pi_1(Y)$, $p: Y \to X$ cover

Claim: Y is a graph

[cover of a graph is a graph]

→ lift the loops to edges of the cover.

Claim: π_1(a graph) is free.

→ contract a max'l tree, to get $\bigvee_\beta S^1$

since $\Gamma \underset{he}{\simeq} \Gamma/\tau$, same fundamental gp

□

K(G,1) Spaces

Def: X is a $K(G,1)$ space if it is path-ctd & $\pi_1(X) = G$, & $\tilde{X}$, univ. cover, is contractible.

ex: • $G = \mathbb{Z}$, $K(G,1) = S^1$ b/c $\tilde{X} = \mathbb{R}$, contractible.

• $G = \mathbb{Z}^2$, $K(G,1) = T^2$ b/c $\tilde{X} = \mathbb{R}^2$, contractible.

• $G = \mathbb{Z}_2$, $X = X_G$; $\tilde{X}_G = S^2$, not contractible,

so $X_G \neq K(G,1)$.

$K(G,1) = \mathbb{R}P^\infty$, b/c $\tilde{X}_G = S^\infty$, contractible

$\mathbb{R}P^\infty \to$ 2-skeleton is $\mathbb{R}P^2$ & so

$\pi_1(\mathbb{R}P^\infty) = \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$

Claim: $\forall G$, there exists a $K(G,1)$ space.

For a fixed G, 2 $K(G,1)$ sp's are h.e., so their topology is really just determined by G, the fund. gp.

11/11 **Homology**

$\pi_1(X) = \{f: S^1 \to X\}/\sim$

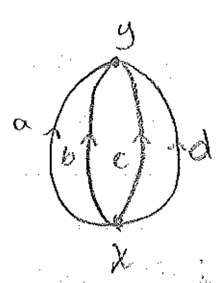
$\pi_n(X) = \{f: S^n \to X\}/\sim$: higher homotopy gps - Ch. 4.
↳ hard to compute.

$H_n(X)$ - homology gp. - easy to calculate

Facts:

- $H_n(X)$ is abelian
- $H_1(X) = Ab(\pi_1(X))$, X path-ctd.
- In CW-complex, $H_n(X)$ depends only on the $(n+1)$-skeleton.
- In general, difficult to define - start w/ def. on special class, Δ-complexes $\subseteq$ CW-complexes
  on Δ-complexes, called simplicial homology.
  on general top. sp, called singular homology.

Ex:



ab^{-1} vs. $b^{-1}a$ - diff. loops in $\pi_1(X)$
$\longleftrightarrow$ $a-b$ - but they trace out the
(write additively) same shape.

Def: • A cycle is a loop without chosen basept. (loosely)
• A chain is a linear combination ($\mathbb{Z}$) of edges
ex: $ka + lb + mc + nd$
• A cycle (precisely) is a chain that leaves & enters each vertex the same # of times
ex: $a - b + c - d = a - d + c - b$

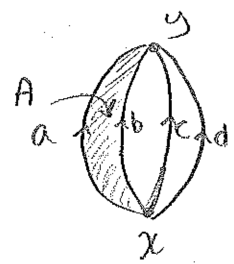
Let C_1 be the free abel. gp w/ basis $\{a, b, c, d\}$, ie the chains.
let C_0 be the free abel. gp w/ basis $\{x, y\}$.

$\partial : C_1 \to C_0$
$a \longmapsto y - x$ (enter y, leave x)
$b \longmapsto y - x$
$c, d \longmapsto y - x$

Note: $\operatorname{Ker}\partial = \langle a-b, b-c, c-d \rangle$ ← each encodes a hole in the space.

Here, $H_1(X) = \operatorname{Ker}\partial_1 \simeq \mathbb{Z}^3$

Ex:



Now, $a-b$ is no longer a hole; it is the boundary of the 2-cell A. To capture this, form the quotient of $\operatorname{Ker}\partial$ by $a-c \sim b-c$

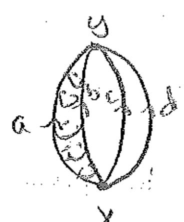
Let C_2 be the free abel. gp w/ basis $\{A\}$.

$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ (both boundary maps)

$\partial_2(A) = a-b$

Here, $H_1(X) = \operatorname{Ker}\partial_1 / \operatorname{Im}\partial_2 \simeq \mathbb{Z}^2$

Ex:



Add 2 2-cells, both to $a-b$, A & B

$C_2 = \langle A, B \rangle$ as abel. gp.

$$\left.\begin{array}{l}\partial_2(A) = a-b \\ \partial_2(B) = a-b\end{array}\right\} \Rightarrow \operatorname{Im}\partial_2 \text{ same as before}$$

$H_1(X) = \operatorname{Ker}\partial_1 / \operatorname{Im}\partial_2 \simeq \mathbb{Z}^2$

Note: $\operatorname{Ker}\partial_2 = \langle A-B \rangle$ } the new higher dim'l hole – the empty sphere

$H_2(X) = \operatorname{Ker}\partial_2 \simeq \mathbb{Z}$

Ex: Fill in the 2-sphere w/ a 3-cell, α

C_3 is the free abel. gp w/ basis $\{\alpha\}$

$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_2$

$\partial_3(\alpha) = A - B$

$H_2 = \operatorname{Ker}\partial_2 / \operatorname{Im}\partial_3 = \langle A-B \rangle / \langle A-B \rangle = 1$ (2-dim'l hole was filled in)

In general, for a cell complex X, $C_n^{(X)}$ is the free abel. gp w/ basis of n-cells. Define a map

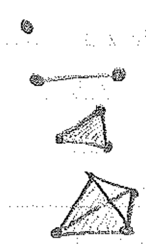
$\partial_n : C_n \to C_{n-1}$ "boundary maps" ($\operatorname{Ker} \subseteq \operatorname{Im}$, b/c the comp. of any 2 is trivial)

Then $H_n(X) = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$

Simplicial Homology - homology for Δ-complexes.
- Δ-complexes are built out of simplices

Def: An n-simplex is the convex hull of $n+1$ points in general position in $\mathbb{R}^m$ (for m large enough).

Ex: 0-simplex: 1-simplex: 2-simplex: 3-simplex: (solid tetrahedron)



Def: A standard n-simplex is $\{(t_0,\dots,t_n)\in\mathbb{R}^{n+1} \mid \sum t_i = 1,\ t_i \geq 0\ \forall i\}$

1-simplex 2-simplex

For homology, we want each simplex to have an ordering on vertices, i.e. $[v_0,\dots,v_n]$ (defines an ordered simplex). This order gives an orientation on edges.

edge $[v_i,v_j]$ is $v_i \longrightarrow v_j$ if $i<j$.

Def: A face of a simplex $[v_0,\dots,v_n]$ is $[v_0,\dots,\hat{v}_i,\dots,v_n]$. (i.e., remove a vertex)
- the order on the simplex gives an order on the faces.

11/13 Constructing Δ-complexes

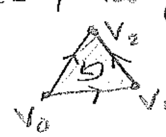
n-simplex: the convex hull of n+1 pts in general position

OR $\Delta^n = \{(t_0, t_1, \ldots, t_n) \mid \sum_i t_i = 1, t_i \geq 0\}$, the standard simplex.

- Orient each simplex: $[v_0, \ldots, v_n]$

Note: • orienting vertices gives an orientation on edges, & on higher dim'l faces

• $[v_0, v_1, v_2]$, edges: $[v_0, v_1], [v_0, v_2], [v_1, v_2]$

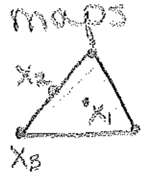


• There's a ! linear isomorphism from the standard n-simplex to any oriented n-simplex that preserves orientation From linear alg & bases.

- Faces: A face of $[v_0, \ldots, v_n]$ is given by $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$. We have an induced orientation on the faces!
- $\partial \Delta^n = \bigcup$ Faces
- open simplex $= \Delta^n - \partial\Delta^n = \mathring{\Delta}^n$

Def: A Δ-complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^{n(\alpha)} \to X$ s.t.

(1) $\sigma_\alpha : \mathring{\Delta}^n \to X$ is injective, and $\forall x \in X$ $\exists!$ map σ_α that maps to x.



$\sigma : \mathring{\Delta}^2 \to x_1$
$\sigma : \mathring{\Delta}^1 \to x_2$
$\sigma : \mathring{\Delta}^0 \to x_3$

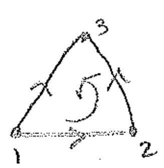
(2) $\sigma_\alpha|_{face} = \sigma_\beta : \Delta^{n-1} \to X$

(3) $A \subseteq X$ is open $\Leftrightarrow$ $\sigma_\alpha^{-1}(A)$ open in the simplex $\forall \alpha$.

Note: (1) Really X is a quotient of a disjoint collection of simplices, where you're identifying faces w/ other simplices.

(2) Or, build X inductively, like a CW-complex, but being careful that orientation is always preserved.

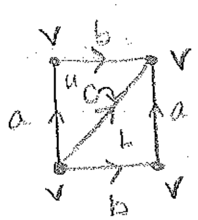
Ex:



Can glue in a 2-cell

can't glue in a 2-cell, b/c it's not an orientation on the vertices.

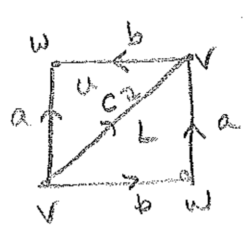
Ex: T



c must be oriented this way to glue in 2 2-cells, U, L.

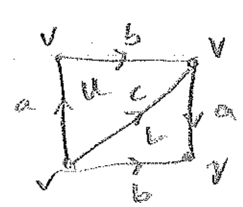
0-simplices: v
1-simplices: a, b, c
2-simplices: U, L

Ex $\mathbb{R}P^2$



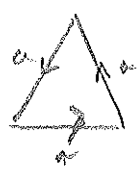
- subdivide each a into a, b

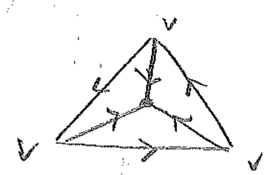
Ex: K



Ex:  $4g$-gon: gives us Σ_g

Ex:  can glue all edges together & get Δ-complex structure (called dunce cap)

 cannot glue all edges together & get a Δ-complex structure.

BUT  is a Δ-complex

Note: • Simplicial complex: A Δ-complex, where each simplex is uniquely determined by its vertices. In particular, each n-simplex has $n+1$ distinct vertices.

• Thm: Each Δ-complex can be subdivided to make a simplicial complex.

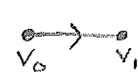
Simplicial Homology

Let X be a Δ-complex.

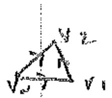
• Define $\Delta_n(X)$ to be the free abelian gp w/ a gen. for each n-simplex, or rather each open n-simplex $\mathring{\Delta}^n$ or equivalently each $\sigma_\alpha : \mathring{\Delta}^n \to X$.

• Define $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$

$\partial_0 : \Delta_0(X) \to 0$

$\partial_1 : \Delta_1(X) \to \Delta_0(X)$, $\partial_1([v_0, v_1]) = [v_1] - [v_0]$ 

$\uparrow$ basis elt in Δ_1

 $\partial_2 : \Delta_2(X) \to \Delta_1(X)$, $\partial_2([v_0, v_1, v_2]) = [v_0, v_1] + [v_1, v_2] - [v_0, v_2]$

 $\partial_3([v_0, v_1, v_2, v_3]) =$

11/15 Simplicial Homology

Δ-complexes: • glue together oriented simplices along faces s.t. the gluing preserves orientation.

OR:

• Maps $\sigma_\alpha : \Delta^n \to X$

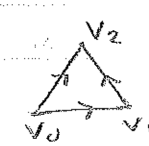
$\Delta_n(X)$ = free abel gp w/ basis all n-simplices.

OR w/ basis $\{\sigma_\alpha \mid \sigma_\alpha : \Delta^n \to X\}$

$\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$

$\partial([v_0, v_1]) = v_1 - v_0$

$\partial([v_0, v_1, v_2]) = [v_0\, v_1] - [v_0 v_2] + [v_1\, v_2]$



$\partial_n([v_0, \dots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0 \cdots \hat{v}_i \cdots v_n]$

and extend by linearity.

OR $\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha |_{[v_0 \cdots \hat{v}_i \cdots v_n]}$

Lemma: $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$.

$\partial_{n-1} \circ \partial_n = 0$.

Pf: $\partial_{n-1} \partial_n([v_0, \cdots v_n]) = \partial_{n-1}\left(\sum_i (-1)^i [v_0 \cdots \hat{v}_i \cdots v_n]\right)$

2 cases: $\sum (-1)^j [v_0 \cdots \hat{v}_j \cdots \hat{v}_i \cdots v_n]$ if j was even, still even

or $\sum (-1)^{j+1} [v_0 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n]$ if j was even, now j odd

same

So $= \sum_{j<i} (-1)^i (-1)^j [v_0 \cdots \hat{v}_j \cdots \hat{v}_i \cdots v_n] + \sum_{j>i} (-1)^i (-1)^{j-1} [v_0 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n]$

$= 0$. $\square$

Note: $\operatorname{Im} \partial_n \subseteq \ker \partial_n$

Def: • A chain complex is a sequence of abel. gps

$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to C_0 \xrightarrow{\partial_0} 0$ w/ maps

$\partial_i : C_i \to C_{i-1}$ s.t. $\partial_n \circ \partial_{n+1} = 0$.

• The n^{th} homology gp is $H_n = \ker \partial_n / \operatorname{Im} \partial_n$

- Elements of $\ker \partial_n$ are called cycles
- Elements of $\operatorname{Im} \partial_{n+1}$ are called boundaries
- Elements of H_n are called homology classes.
- Two cycles are homologous if they represent the same homology class.

Ex: Simplicial Homology

$C_n = \Delta_n(X)$, X a Δ-complex

∂_n as before.

This gives a chain complex.

$H_n = \underbrace{H_n^{\Delta}(X)}_{\text{notation}} = \ker \partial_n / \operatorname{Im} \partial_{n+1}$

Ex: $X = S^1$  [Glue the 2 ends of a 1 simplex together]

$$\cdots \to 0 \xrightarrow{\partial_3} \underset{\substack{\text{no 2} \\ \text{simplices}}}{0} \xrightarrow{\partial_2} \underset{C_1}{\langle e \rangle} \xrightarrow{\partial_1} \underset{C_0}{\langle v \rangle} \xrightarrow{\partial_0} 0$$

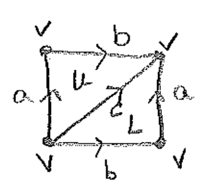
$\partial_0 = 0$, $\partial_1 : e \mapsto v - v = 0 \Rightarrow \partial_1 = 0$, $\partial_2 =$ inclusion $= 0$

$H_0^{\Delta}(X) = \ker \partial_0 / \operatorname{Im} \partial_1 = \langle v \rangle / 0 = \langle v \rangle \simeq \mathbb{Z}$

$H_1^{\Delta}(X) = \ker \partial_1 / \operatorname{Im} \partial_2 = \langle e \rangle / 0 = \langle e \rangle \simeq \mathbb{Z}$

$H_2^{\Delta}(X) = \ker \partial_2 / \operatorname{Im} \partial_3 = 0/0 = 0$

$H_n^{\Delta}(X) = 0 \quad \forall n \geq 2$

Ex: $X = T^2$ 

$$\cdots 0 \to \underset{C_3}{0} \xrightarrow{\partial_3 = 0} \langle U, L \rangle \xrightarrow[\substack{U \mapsto a+b-c \\ L \mapsto b+a-c}]{\partial_2} \underset{C_1}{\langle a,b,c \rangle} \xrightarrow[\substack{a,b,c \mapsto v-v}]{\partial_1 = 0} \underset{C_0}{\langle v \rangle} \xrightarrow{\partial_0 = 0} 0$$

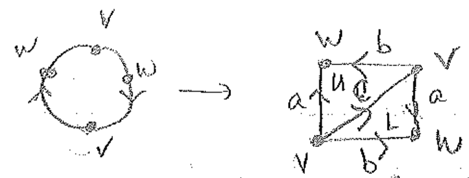
$H_0^{\Delta}(X) \simeq \mathbb{Z}$

$H_1^{\Delta}(X) = \langle a,b,c \rangle / \langle a+b-c \rangle \simeq \mathbb{Z} \oplus \mathbb{Z} \simeq \mathbb{Z}^2$

$H_2^{\Delta}(X) = \langle U - L \rangle / 0 = \langle U - L \rangle \simeq \mathbb{Z}$

$H_n^{\Delta}(X) = 0 \quad \forall n \geq 3$

Ex: $X = \mathbb{R}P^2$



injective (map from $\mathbb{Z}^2$ & image has rank 2)

$$C_3 = 0 \xrightarrow{\partial_3 = 0} \langle u, L\rangle \xrightarrow{\partial_2} \langle a, b, c\rangle \xrightarrow{\partial_1} \langle v, w\rangle \xrightarrow{\partial_0 = 0} 0$$

$u \mapsto c+b-a$ $\quad$ $a \mapsto w-v$

$L \mapsto c+a-b$ $\quad$ $b \mapsto w-v$

$\quad$ $c \mapsto 0$

$H_n^\Delta(X) = 0 \quad n \geq 3$

$H_2^\Delta(X) = \ker\partial_2 / \operatorname{Im}\partial_3 = 0$

$H_1^\Delta(X) = \ker\partial_1 / \operatorname{Im}\partial_2 = \langle a-b, c\rangle / \langle c+b-a, c+a-b\rangle = \langle c, a-b+c\rangle / \langle c+a-b, 2c\rangle$ sum of 2 gens

$= \mathbb{Z}_2$

$H_0^\Delta(X) = \langle v, w\rangle / \langle w-v\rangle \cong \mathbb{Z}$

Do Klein bottle!

Questions:

- Are $H_n(X)$ independent of the Δ-complex structure on X?
- If $X \underset{he}{\simeq} Y$, is $H_n^\Delta(X) = H_n^\Delta(Y)$?
- What about non-Δ-complex structure?

Singular Homology

Look at all possible maps $\sigma: \Delta^n \to X$ w/ no restrictions.

$C_n(X) :=$ free abel. gp on all $\{\sigma \mid \sigma: \Delta^n \to X\}$.

11/18 $\quad$ $\partial_n: C_n(X) \to C_{n-1}(X)$, restricting σ to what happens on the boundary.

$$\partial_n(\sigma) = \sum_{i=1}^{n} \sigma|_{[v_0 \ldots \hat{v}_i \ldots v_n]} (-1)^i, \quad \sigma: [v_0 \cdots v_n] \to X$$

Singular homology: $H_n(X) = \ker\partial_n / \operatorname{Im}\partial_{n+1}$

Notes: ① Homeomorphic spaces have isomorphic sing. homology gps.

② $C_n(X)$ usually have an uncountable basis.

③ For all X, $\exists\, S(X)$, a delta complex s.t.

$H_n(X) \cong H_n^\Delta(S(X))$

$S(X) = \sqcup\, \sigma / \sim$, where $\sim$ is gluing along faces s.t.

$\sigma_1|_{\text{face } 1} = \sigma_2|_{\text{face}}$

$S(X)$ is not homeo. to X, but it is a Δ-complex (albeit huge & ugly)

Geometric Interpretation:

$\sigma_1 : \Delta \longrightarrow X$

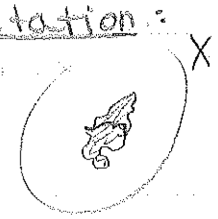
$\sigma_2 : \Delta \longrightarrow X$

$\Downarrow$

$f : \Diamond \longrightarrow X$

K

Δ-complex



(they agree, in an orientation-preserving way, on an edge)

④ Each n-chain ξ can be written as
$\xi = \sum n_i \sigma_i^n$, wlog $n_i = \pm 1$ (ie. allow repeats)
(finite sum)

$\partial \xi = \sum n_j \sigma_j^{n-1}$, $n_j = \pm 1$

we might have cancelling pairs (as in f, above) of σ_j^{n-1} $\rightarrow$ which implies those 2 edges were identified

⑤ By identifying σ_i^n along faces that corresp to cancelling pairs, get $K_\xi \rightarrow X$, K_ξ a Δ-complex

What if ξ is a cycle? (ie. $\partial\xi = 0$, so everything is a cancelling pair, so all faces were glued, so get a manifold, ie K_ξ is an oriented manifold (along $(n-1)$-faces), away from the $(n-2)$-subcomplex.

e.g. If $n=2$, get a closed surface

e.g. If $n=1$, get a loop (ie, edges whose endpts are all glued together)

⑥ A one-cycle in $H_1(X)$ represents the zero class (i.e. is in ker) if it extends to a map of an oriented surface, i.e. is the boundary of an oriented surface.

Prop: If $X = \sqcup X_\alpha$, X_α path-components, then
$H_n(X) \simeq \oplus_\alpha H_n(X_\alpha)$

Pf: $C_n(X) \simeq \oplus_\alpha C_n(X_\alpha)$

∂_n preserves ↗ (ie $\partial_n C_n(X_\alpha) \in C_{n-1}(X_\alpha)$)
since σ cts & Δ ctd, so image is in a particular path-component. □

Prop: If $X \neq \emptyset$ & path-ctd, then $H_0(X) \simeq \mathbb{Z}$

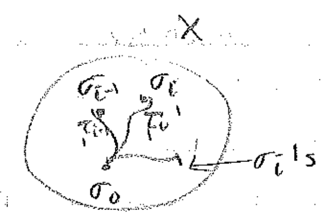
Pf: $H_0(X) = \operatorname{Ker}\partial_0 / \operatorname{Im}\partial_1$ $\qquad C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$

define $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$
$\sum n_i \sigma_i \longmapsto \sum n_i$ $\qquad$ (ε surj b/c can get 1)
↑ map of bunch of pts into X

Claim: $\operatorname{Ker}\varepsilon = \operatorname{Im}\partial_1$ if X is path-ctd

Suppose $\sum n_i \sigma_i \in \operatorname{Ker}\varepsilon$, ie $\sum n_i = 0$. Need to construct a 1-chain whose boundary is $\sum n_i \sigma_i$

Connect σ_0 to σ_i by path τ_i.
τ_j are singular 1-chains.



A 1-chain is just a path

$\partial(\sum n_i \tau_i) = \sum n_i(\sigma_i - \sigma_0) = \sum n_i \sigma_i - \underbrace{\sum n_i \sigma_0}_{=0} \Rightarrow \sum n_i \sigma_i$ is a boundary
$\Rightarrow \operatorname{Ker}\varepsilon \subseteq \operatorname{Im}\partial_1$

Conversely, $\varepsilon(\partial(\sigma)) = \varepsilon(v_1 - v_0) = 0$
(extend to sum)

Then $\mathbb{Z} \simeq C_0(X)/\operatorname{Ker}\varepsilon = C_0(X)/\operatorname{Im}\partial_1 = \operatorname{Ker}\partial_0/\operatorname{Im}\partial_1$ □.

Prop: If $X = *$, then $H_n(X) = 0$ for $n > 0$, & $H_0(X) = \mathbb{Z}$.

Pf: $C_n(X) \simeq \mathbb{Z}$, b/c gen. by the unique map $\sigma_n : \Delta^n \longrightarrow *$.

$$\partial_n(\sigma_n) = \sum (-1)^i \sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]} = \sum (-1)^i \sigma_{n-1} = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sigma_{n-1} & \text{if } n \text{ even} \end{cases}$$

$$\cdots \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\simeq} \underset{\substack{\uparrow \\ n}}{\mathbb{Z}} \xrightarrow{0} \mathbb{Z} \xrightarrow{\simeq} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} 0.$$

$H_n(X) = \mathbb{Z}/\mathbb{Z} = 0$

$H_{n+1}(X) = 0/0$ b/c $\ker = 0$ & $\operatorname{Im} = 0$. □

11/20 <u>Homology vs. Fundamental Gp</u>

$f: I \to X$ can be thought of as a path, or as a singular 1-simplex

- If f is a loop, it defines an elt in $\pi_1(X, x_0)$ or it defines a 1-cycle: $(f(0) = f(1) = x_0)$

  $\partial_1(f) = f(0) - f(1) = 0$ $\qquad (H_1(X) = \underset{\text{cycles}}{\ker \partial_1} / \underset{\text{boundaries}}{\operatorname{Im} \partial_2})$

  (i.e, f represents a class in the 1st Hom. Gp)

<u>Thm</u>: $h: \pi_1(X, x_0) \to H_1(X)$ is a surj. homomorphism
(X path-ctd) $\quad \underset{\text{homotopy class}}{[f]} \longmapsto \underset{\text{homology class}}{[f]} = f \cdot \operatorname{Im}\partial_2 \leftarrow f$ coset of $\operatorname{Im}\partial_2$

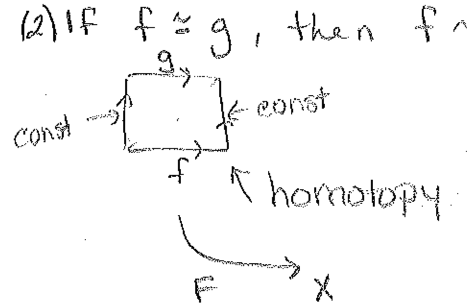
and $\ker h = [\pi_1(X,x_0), \pi_1(X,x_0)]$, i.e. $H_1(X) = \pi_1(X,x_0)/\pi_1(X,x_0)'$
$= \pi_1(X,x_0)_{ab}$, the abelianization.

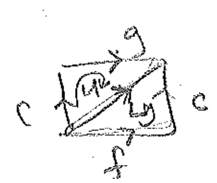
<u>Pf</u>: $\quad f \simeq g$: path homotopic
$\qquad f \sim g$: homologous (i.e. $f - g$ is a boundary)

<u>Facts</u>:

(1) If f is constant, $f \sim 0 \Rightarrow f$ is a boundary
$(f: I \to x_0)$ [$\sigma: \Delta^2 \to x_0 \in X$, $\partial\sigma = f + f - f = f$]
$\qquad \uparrow$ constant 2-simplex

(2) If $f \simeq g$, then $f \sim g$



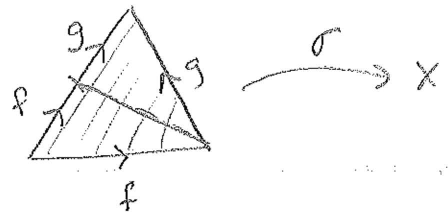


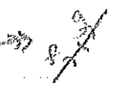
$F|_u = \sigma_1$
$F|_L = \sigma_2$

$\partial(\sigma_1 - \sigma_2) = f - g + c + c$ (const) ✓
opp. orientations $\qquad$ boundaries by 1

Really need $\partial(\sigma_1 - \sigma_2 - \sigma_3 - \sigma_4)$, where $\sigma_{3,4}$ are const 2 simplices.

(3) $f * g \sim f + g$



σ : project $\Delta \twoheadrightarrow$  then map to X.

$\partial\sigma = f + g - (f * g)$.

(4) $\bar{f} \sim -f$ since $\underbrace{\bar{f} * f}_{\text{const. map } \sim 0} \sim \bar{f} + f$ by (3)

$\Rightarrow \bar{f} + f \sim 0 \Rightarrow \bar{f} \sim -f$.

Thus, h is a well-defined homomorphism, by (2) & (3).

h is surjective:

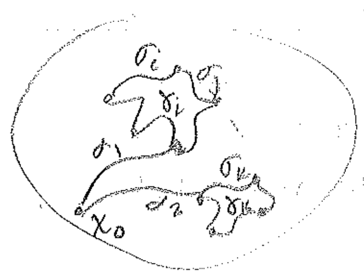
Let $\sum n_i\sigma_i$ be a 1-cycle, $\sigma : I \to X$

Wlog, $n_i = \pm 1$, wlog $n_i = 1$ (if $n_i = -1$, replace σ w/ $\bar{\sigma}$). Have $\sum \sigma_i$, & know $\partial(\sum \sigma_i) = 0$

$\Rightarrow \sum \gamma_i$. Get a loop $\gamma = *_i\, \alpha_i \gamma_i \bar{\alpha}_i$

$h(\gamma) = \sum \gamma_i + \sum \alpha_i + \sum \bar{\alpha}_i$

$= \sum \gamma_i = \sum \sigma_j$ ✓



Ker $h = [\pi_1(X, x_0), \pi_1(X, x_0)]$:

(1) $[\pi_1(X, x_0), \pi_1(X, x_0)] \subseteq$ Ker h b/c $H_1(X)$ is abel.

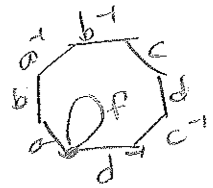
(2) Ker $h \subseteq [\ ,\ ]$:

Take $f \in$ Ker h, show $[f] = 0$ in $\pi_1(X, x_0)_{ab}$

$f = \partial(\sum n_i \sigma_i)$

↑ a 2-chain :  $\longrightarrow X$

gluing Δ's together gives an orientable surface w/ boundary f

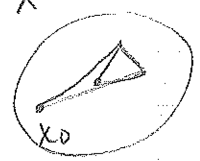
But orientable surfaces are $4g$-gons, whose boundaries are commutator eg.  $\Rightarrow f \simeq [a,b][c,d]$

So $f \simeq$ product of commutators
$\Rightarrow f = 0$ in the abelianization

OR

$K =$  (complex after gluing σ_i's together)

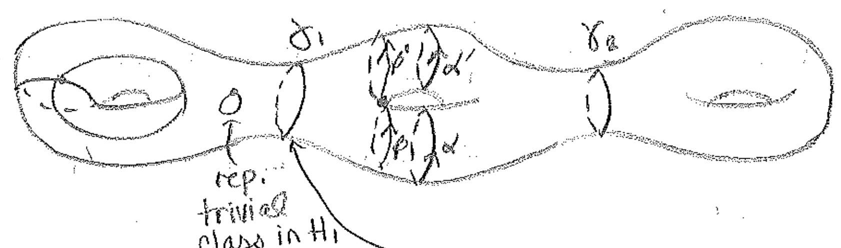
$\xrightarrow{\sigma_i} X$



- homotope Δ's to have a vertex x_0
- each Δ is null-hom, b/c $\sigma_i \in \ker h$.

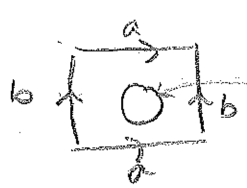
ex: read rest...

Ex: 1st homology of a surface, Σ_g



$\gamma_1 \sim \gamma_2 \sim 0$

commutator b/c bounds surface (left or right)



$\alpha_1 + \alpha_2 \sim 0$ b/c

$0 \sim \gamma \simeq \beta * \beta' \sim \beta + \beta' \Rightarrow \beta \sim -\beta'$

11/22 <u>Reduced Homology</u>

Usual chain complex: $\cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{0} 0$

<u>Augmented chain complex</u>:

$$\cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{0} 0$$

$\varepsilon(\sum n_i\sigma_i) = \sum_i n_i$

check: $\partial_1\varepsilon = 0$

<u>Def</u>: <u>reduced homology</u> is homology of augmented chain complex, $\tilde{H}_n(X)$.

Note: $\tilde{H}_n(X) = H_n(X)$ if $n \neq 0$.

recall: $H_0(X)$ = path components of X = $C_0(X)/\mathrm{Im}\,\partial_1$

$\tilde{H}_0(X) = \mathrm{Ker}\,\varepsilon/\mathrm{Im}\,\partial_1$

$* \quad H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$.

i.e. If $X = \{*\}$, have all zero homology gps

<u>Homotopy Invariance</u> (If $X \underset{h.e.}{\simeq} Y$, then $H_n(X) \cong H_n(Y)$)

Want $f: X \to Y$ induces $f_*: H_n(X) \to H_n(Y)$ (like for π_1)

<u>Note</u>: • $f: X \to Y$ induces a map on a single simplex $\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y$ into Y, & then extend by linearity.

so get $f_\#: C_n(X) \to C_n(Y)$.

• $f_\#\partial(\sigma) = f_\#\left(\sum_i (-1)^i \sigma|_{[v_0 \cdots \hat{v}_i \cdots v_n]}\right) = \sum_i (-1)^i (f\circ\sigma)|_{[v_0 \cdots \hat{v}_i \cdots v_n]} =$

$\partial(f_\#\sigma)$:

$$\begin{array}{ccccccccc} C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & C_{n-2}(X) & \longrightarrow & \\ \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & & \\ C_{n+1}(Y) & \longrightarrow & C_n(Y) & \longrightarrow & C_{n-1}(Y) & \longrightarrow & C_{n-2}(Y) & \longrightarrow & \end{array}$$

$\leftarrow$ this diagram commutes.

<u>Def</u>: $f_\#$ is called a <u>chain map</u>, i.e. $f_\#: C_n(X) \to C_n(Y)$ & $f_\#\partial = \partial f_\#$.

- $f_\#$ takes cycles to cycles.
  If σ cycle, $\partial(\sigma)=0 \Rightarrow f_\# \partial(\sigma)=0 \Rightarrow \partial(f_\# \sigma)=0$.
- $f_\#$ takes boundaries to boundaries (ie if σ_n a bound, can be written as $\partial(\sigma_{n+1})$)
  $\partial(\sigma_{n+1})$ is a boundary $\Rightarrow$ $\partial(f_\# \sigma_{n+1}) = \sigma_n$ a boundary?
- So $f_\#$ induces a map $f_*: H_n(X) \to H_n(Y)$
  (b/c takes cosets of boundary gp to cosets of bound. gp)
  & f_* a homomorphism, by linearity.

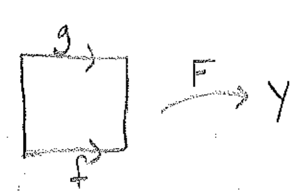
<u>Prop</u>: Any chain map btwn chain complexes induces a homomorphism btwn homology gps.
<u>Pf</u>: We just did it.

<u>Properties</u>:
- $(fg)_* = f_* g_*$
- $Id_* = Id$

<u>Thm</u>: If $f,g: X \to Y$ s.t. $f \simeq_{he} g$, then $f_* = g_*: H_n(X) \to H_n(Y)$.

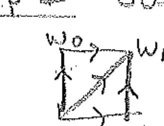
<u>Cor</u>: If $X \simeq_{he} Y$, then $H_n(X) \cong H_n(Y)$
<u>Pf</u>: Take $X \xrightarrow{f} Y$ a homotopy equiv. Then $\exists\, g: Y \to X$ s.t. $f \circ g \simeq_{he} \mathbb{1}_X$, but then $(f \circ g)_* = (\mathbb{1}_X)_* = \mathbb{1}$ & w/ other comp $\Rightarrow$ have f_*, g_* inverses. $\square$

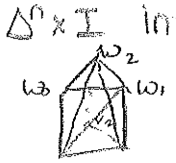
<u>Pf</u> (of Thm):
<u>ex</u>: $X = I$

$\Delta^1 \times I$ 

$[v_0 v_1 v_2 \ldots$

<u>Step 1</u>: Subdivide $\Delta^n \times I$ into simplices





$\cdots$ $\begin{matrix} [w_0, \ldots, w_n] \\ | \quad\quad | \\ [v_0, \ldots, v_n] \end{matrix}$, Then

$[v_0 v_1 w_1]$ & $[v_0 w_0 w_1]$

Simplices of $\underbrace{\Delta^n \times I}_{(n+1)\text{ dim}}$ are $\underbrace{[v_0 \cdots v_i w_i \cdots w_n]}_{n+2 \text{ vertices}}$

Step 2: Given a homotopy $F: X \times I \to Y$ from f to g define $P: C_n(X) \to C_{n+1}(Y)$, (a prism operator) s.t.

$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times Id)\big|_{[v_0 \cdots v_i w_i \cdots w_n]}$

so $k = \Delta^n \times I \xrightarrow{\sigma \times Id} X \times I \xrightarrow{F} Y$

$\Delta^n \xrightarrow[\sigma]{} X$

Step 3: (Claim) $\partial P = g_\# - f_\# - P\partial$

induced map on chains: $\partial P: C_n(X) \xrightarrow{P} C_{n+1}(Y) \xrightarrow{\partial_{n+1}} C_n(Y)$

boundary of whole prism = upper bound − lower bound − lateral sides

Pf: $\partial P(\sigma) = \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times Id)\big|_{[v_0 \cdots \hat{v}_j \cdots v_i w_i \cdots w_n]}$ +

$\sum_{j \ge i} (-1)^i (-1)^{j+1} F \circ (\sigma \times Id)\big|_{[v_0 \cdots v_i w_i \cdots \hat{w}_j \cdots w_n]}$

- when $i=j$, all terms cancel except

→ things cancel....

11/26

Tue	12/3	5-7	problem session
Mon	12/9		Final Exam − in class
Fri	12/14		No class

Note: If $\alpha \in C_n(X)$ a cycle, then

$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P(\partial\alpha) = \partial P(\alpha)$, a boundary.

$\Rightarrow g_\# \sim f_\#$. □

Def: $P: C_n(X) \to C_{n+1}(Y)$ is a chain homotopy btwn maps $f_\#$ & $g_\#$ (if P sat $\partial P = g_\# - f_\# - P\partial$).

Prop: Chain homotopic maps induce the same map on homology.

Exact Sequences & Excision

Goal: relate $H_n(X)$, $H_n(A)$, & $H_n(X/A)$ for $A \subseteq X$.

Def: • A long exact sequence is a seq,
$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$,
s.t. $\mathrm{Im}(\alpha_{n+1}) = \ker(\alpha_n)$.
(if A_i are abelian, this is a chain complex & all H_n's $= 0$).

• A short exact sequence is $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ w/ $\mathrm{Im}\,\alpha = \ker\beta$.
$\Rightarrow \beta$ surj, α inj.

Goal Thm: X, $A \subseteq X$, $A \neq \emptyset$, closed, & a def. retr. of a nbhd of A in X. Then $\exists$ a l.e.s.
$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots \to \tilde{H}_0(X/A) \to 0$$
$i: A \to X$ $\quad j: X \to X/A$

Def: Any pair (X, A) that sat. hyp. of goal thm is called a good pair.
• (X, A) a CW pair is a good pair

Cor: $\tilde{H}_n(S^n) \simeq \mathbb{Z}$, & $\tilde{H}_i(S^n) = 0$, $i \neq n$.
Pf: $X = D^n$, $A = S^{n-1} = \partial D^n$, $X/A = S^n$
$$\cdots \to \tilde{H}_i(S^{n-1}) \to \underset{0}{\tilde{H}_i(D^n)} \to \tilde{H}_i(S^n) \to \tilde{H}_{i-1}(S^{n-1}) \to \underset{0}{\tilde{H}_{i-1}(D^n)}$$
D^n contr $\Rightarrow$ h.e. to a pt $\Rightarrow \tilde{H}_i(D^n) = 0$ inj. & surj $\Rightarrow$ iso.

$\Rightarrow \tilde{H}_n(S^n) \simeq \tilde{H}_{n-1}(S^{n-1}) \simeq \cdots$

ex: $H_n(S^n) \simeq \mathbb{Z}$ (use simplicial homology)

$\tilde{H}_i(S^n) \simeq \tilde{H}_{i-1}(S^{n-1}) \simeq \cdots$
Note: $\tilde{H}_0(S^n) = 0$, use induction to get always to $i \neq n$.

Instead of $\tilde{H}_n(X/A)$, we'll look at $H_n(X,A)$, where $H_n(X,A)$ are the relative homology gps.

Def: Given X, A any subspace, $C_n(X,A) = C_n(X)/C_n(A)$

note, $\partial: C_n(X) \to C_{n-1}(X)$ sends $\partial(C_n(A)) \subseteq C_{n-1}(A)$, b/c boundary of simplicial complex is still in A.

so ∂ induces a map $\partial: C_n(X,A) \to C_{n-1}(X,A)$. &

$\partial \circ \partial = 0$ b/c true for orig maps.

Thus $H_n(X,A)$ is the homology of this chain complex. $C_n(X,A)$

- $H_n(X,A)$ is represented by relative cycles: n-chains α s.t. $\partial\alpha \in C_{n-1}(A)$.
- A relative cycle α is trivial in $H_n(X,A) \Leftrightarrow$ $\alpha = \partial\beta + \gamma$, where $\beta \in C_{n+1}(X)$ & $\gamma \in C_n(A)$. Called a relative boundary.

Thm: Given X, A, $\exists$ l.e.s

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \cdots$$

Pf: By homological algebra.

Lemma: Given

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i_*"] \arrow[d, "\partial"] & C_n(X) \arrow[r, "j_*"] \arrow[d, "\partial"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i_*"] & C_{n-1}(X) \arrow[r, "j_*"] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

The diagram commutes

11/27 <u>Goal</u>: Use the SES in lemma to get a LES

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

In general, given a SES of chain complexes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & C_{n+1} \ni c' \arrow[d] & \\
0 \arrow[r] & A_n \arrow[r, hook, "i"] \arrow[d, "\partial"] & B_n \ni b \arrow[r, two heads, "j"] \arrow[d] & C_n \ni c, \partial c' \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i"] \arrow[d] & B_{n-1} \ni \partial b \arrow[r, "j"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Goal: Take cycle in C_n & get cycle in A_{n-1}, well-def up to homology.

- Take $c \in C_n$ a cycle, i.e. $\partial c = 0$, $c \in \ker \partial$
  - $\exists\, b \in B_n$ s.t. $j(b) = c$ (not unique)
  - $\partial b \in \ker j$ b/c $\partial c = 0$ & right square commutes
  - $\exists!\, a \in A_{n-1}$ s.t. $i(a) = \partial b$
- If $b' \in B_n$ s.t. $j(b') = c$, then $b' - b \in \ker j \Rightarrow \exists\, a' \in A_n$ s.t. $i(a') = b' - b \Rightarrow b' = b + i(a')$
  $\Rightarrow \partial b' = \partial b + \partial i(a') = \partial b + i(\partial a') = i(a) + i(\partial a')$
  $= i(a + \partial a')$, so the guy that maps onto b' differs from a by a boundary, so $a \sim a + \partial a'$.
- If I replace c w/ $c + \partial c'$ [i.e. $c \sim c + \partial c'$]
  Then $\exists\, b' \in B_{n+1}$ s.t. $c' = j(b')$, so
  $c + \partial c' = c + \partial j(b') = c + j(\partial b') = j(b + \partial b')$,
  $\partial(b + \partial b') = \partial b + 0$, so get to same place in B_{n-1} ✓

Thus we have a well-def. map from
$H_n(C) \to H_{n-1}(A)$
$[c] \mapsto [a]$ (a gotten as above)

Claim: This is a homomorphism.
Pf: Everything is linear. ✓

Thm: $\cdots \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to \cdots$

is exact.

Pf: read / exercise.

$$H_{n+1}(X,A) \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A)$$

Note: If $H_n(X,A)=0$ $\forall n$, then $H_n(A) \cong H_n(X)$.

Ex: $(X,A) = (D^n, \partial D^n)$, $\partial D^n = S^{n-1}$. $H_i(D^n) = \begin{cases} 0 & \forall i \neq n, \text{ b/c } D^n \text{ contractible} \\ \mathbb{Z} & \text{if } n=i,\ i\neq 0 \end{cases}$

Note: relative homology at $i=0$ is 0 (not $\mathbb{Z}$) b/c get $\mathbb{Z}/\mathbb{Z}$.

$$0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$$

$$\vdots$$

$$\begin{array}{ccccccccc} 0 & \to & C_0(A) & \to & C_0(X) & \to & C_0(X,A) & \to & 0 \\ & & \downarrow \varepsilon & & \downarrow \varepsilon & & \downarrow & & \\ 0 & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & 0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

path-cted.

$\Rightarrow$ get $\tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X,A)$

i.e. rel. homology is same as reduced hom. $\tilde{H}_n(X,A) \to H_n(X,A)$ $\forall n$ (including 0)

- By same argument, get LES of reduced homs

$$\tilde{H}_i(D^n) = 0 \ \forall i \Rightarrow H_i(D^n, \partial D^n) \cong \tilde{H}_{i-1}(\partial D^n) = \begin{cases} \mathbb{Z} & \text{if } i=n \\ 0 & \text{if } i\neq n. \end{cases}$$

Ex: (X, x_0)
$\tilde{H}_n(x_0) = 0 \ \forall n \Rightarrow H_n(X, x_0) \cong \tilde{H}_n(X)$.

Note: Given a map $f: (X,A) \to (Y,B)$, this induces a map $f_\#: C_n(X,A) \to C_n(Y,B)$

$$\underset{C_n(X)/C_n(A)}{C_n(X,A)} \qquad \underset{C_n(Y)/C_n(B)}{C_n(Y,B)}$$

(well-def b/c a chain in A mapped by f to a chain in B)

Claim: $f_\# \partial = \partial f_\#$ (exercise)
So $f_\#$ induces a map $f_*: H_n(X,A) \to H_n(Y,B)$

Prop If $f,g: (X,A) \to (Y,B)$ h.e. via a homotopy s.t. $f_t: (X,A) \to (Y,B)$, then $f_* = g_*: H_n(X,A) \to H_n(Y,B)$.
Pf: Same as before, but check that everything works w/ A & B.

Generalize: (X,A,B), $B \subseteq A \subseteq X$. Then $\exists$ SES of chains

$$0 \to \underset{C_n(A)/C_n(B)}{C_n(A,B)} \xrightarrow{i} \underset{C_n(X)/C_n(B)}{C_n(X,B)} \xrightarrow{j} \underset{C_n(X)/C_n(A)}{C_n(X,A)} \to 0$$

inclusion quotienting

So $\exists$ LES
$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \cdots$$

12/2 Ex: $H_n(X, x_0) \Rightarrow H_n(X, x_0) \cong \tilde{H}_n(X)$. (b/c $\tilde{H}_n(x_0) = 0 \ \forall n$)

$\boxed{\text{Excision}}$

Thm: The two statements are equiv. & true:
(1) $Z \subseteq A \subseteq X$, & $\bar{Z} \subseteq A^\circ$ ($A^\circ = \operatorname{int} A$), then
$i : (X - Z, A - Z) \to (X, A)$ induces an isomorphism
$i_* : H_n(X - Z, A - Z) \to H_n(X, A)$.
(2) If $A, B \subseteq X$ s.t. $A^\circ \cup B^\circ = X$, then $i : (B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism $i_* : H_n(B, A \cap B) \to H_n(X, A)$.

ex: Show 2 formulations are equiv.
Pf: (read)

Recall: Initial goal was to compare $H_n(A)$, $H_n(X)$, & $H_n(X/A)$.

Thm: $\exists$ LES $\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \cdots$
if (X, A) is a good pair (ie. A is a def. retr. of some nbhd in X).

Prop: For good pairs (X, A), $q : (X, A) \to (X/A, A/A)$ induces an isomorphism $q_* : H_n(X, A) \to H_n(X/A, \underset{\uparrow pt}{A/A}) \cong \tilde{H}_n(X/A)$
Pf: Let V = nbhd that def. retr's onto A, (X, V, A). Then diagram commutes

$$\begin{tikzcd}
H_n(X,A) \arrow[r, hook, "@"', "\cong"] \arrow[d, "q_*"] & H_n(X,V) \arrow[d, "q_*"] & H_n(X-A, V-A) \arrow[l, "*"'] \arrow[d, "q_*\ \#"] \\
H_n(X/A, A/A) \arrow[r, hook, "@"', "\cong"] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "*"']
\end{tikzcd}$$

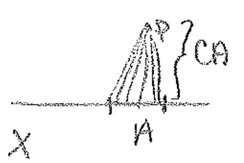
@ is an iso by LES for triples:
$$\cdots \to \underset{\substack{\cong \\ 0}}{H_n(V,A)} \to H_n(X,A) \to H_n(X,V) \to \underset{\substack{\cong \\ 0}}{H_{n-1}(V,A)} \to \cdots$$
b/c A def. retr. of V

$\circledast$ is an isomorphism from excision b/c $\bar{A} \subseteq V^\circ$.
$\circledast\!\!\circledast$ is an isomorphism b/c q is a homeomorphism off A (b/c quotient) & homeo's induce iso's on H_n.

By commutativity, center q_* an iso, & then again by comm, left q_* an isom. $\square$

Thus the thm follows.

<u>Note</u>: If (X,A) is not a good pair, we can still view $H_n(X,A)$ as the reduced homology of $X \cup CA$



$$\tilde{H}_n(X \cup CA) \underset{\substack{\uparrow \\ \text{LES of } (X, CA) \text{ for } \tilde{H}_n \\ \text{\& } CA \text{ retr.}}}{\cong} H_n(X \cup CA, CA) \underset{\substack{\uparrow \\ \text{excision}}}{\cong} H_n(X \cup CA - p, CA - p) \underset{\substack{\uparrow \\ \text{b/c } X\cup CA - p \text{ def retr onto } X \text{ \& } \\ CA - p \text{ def retr onto } A.}}{\cong} H_n(X,A)$$

<u>Cor</u>: If X is a CW complex, $X = A \cup B$, w/ A & B subcomplexes, then $(B, A\cap B) \hookrightarrow (X,A)$ induces an isomorphism on homology, $H_n(B, A\cap B) \to H_n(X,A)$

<u>Pf</u>: excision: $B/A\cap B \underset{\text{homeo}}{\cong} X/A$ so $\tilde{H}_n(B/A\cap B) \cong \tilde{H}_n(X/A) \cong H_n(X,A)$

$\tilde{H}_n(B/A\cap B) \cong H_n(B, A\cap B)$ b/c good pairs

<u>Cor</u>: $\vee_\alpha X_\alpha$, then $i_\alpha : \tilde{H}_n(X_\alpha) \to \tilde{H}_n(\vee_\alpha X_\alpha)$ induces an iso. $\oplus i_\alpha : \oplus \tilde{H}_n(X_\alpha) \to \tilde{H}_n(\vee_\alpha X_\alpha)$ provided wedge is formed at basepts x_α s.t. (X_α, x_α) a good pair.

<u>Pf</u>: ex.

Classical Result:

Thm: If $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ open$^{\neq \emptyset}$, & $U \cong V$ (homeo), then $n = m$.

Pf: $H_k(U, U - \{x_0\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m - \{x_0\})$ by excision

$$\cong \text{ by LES}$$

$$\tilde{H}_{k-1}(\mathbb{R}^m - \{x_0\})$$

$$\cong$$

$$\tilde{H}_{k-1}(S^{m-1}) \cong \begin{cases} \mathbb{Z} & m = k \\ 0 & \text{else} \end{cases}$$

Similarly, $H_k(V, V - \{y_0\}) \cong \begin{cases} \mathbb{Z} & n = k \\ 0 & \text{else} \end{cases}$

But $U \cong V \Rightarrow H_k(U, U - \{x_0\}) \cong H_k(V, V - \{y_0\}) \Rightarrow n = m$. □

12/3 $\boxed{\text{Applications}}$

Degree of a map $f: S^n \to S^n$

f induces a map $f_*: H_n(S^n) \to H_n(S^n)$

$$\begin{array}{ccc} H_n(S^n) & \longrightarrow & H_n(S^n) \\ \cong & & \cong \\ \mathbb{Z} & & \mathbb{Z} \\ 1 & \longmapsto & d \end{array}$$

Def: d is the degree of f.

Properties:

(a) $\deg \mathrm{Id} = 1$ (b/c Id_* is an iso, so $1 \mapsto 1$)

(b) If f is not surjective, $\deg f = 0$ (since $f: S^n \to S^n \setminus \{y\} \hookrightarrow S^n$

so $f_*: H_n(S^n) \to \underset{0}{H_n(S^n \setminus \{y\})} \to H_n(S^n)$ b/c $S^n \setminus \{y\}$ is contractible)

(c) If $f \simeq_{he} g$, $\deg g = \deg f$

(c') If $f: S^n \to S^n$ is a h.e., then $\deg f = \pm 1$, since $\exists g$ s.t. $f \circ g \simeq_{he} \mathbb{1} \Rightarrow \deg f \cdot \deg g = 1$

(d) $\deg f \circ g = \deg f \cdot \deg g$ b/c $(f \circ g)_* = f_* \circ g_*$

(e) $\deg f = -1$ if f is a reflection (i.e. fix $S^{n-1} \subseteq S^n$, & exchange hemispheres)

Take a Δ-complex structure on S^n w/ Δ_1^n & Δ_2^n glued together. homology gen. by $\Delta_1^n - \Delta_2^n$

$$\Delta_1^n - \Delta_2^n \overset{f}{\longmapsto} -(\Delta_1^n - \Delta_2^n) = \Delta_2^n - \Delta_1^n$$

(f) If f is the antipodal map $-\mathrm{Id}: S^n \to S^n$, $\deg f = (-1)^{n+1}$ since the antip. map is a composition of reflections in each coord, & $S^n \subseteq \mathbb{R}^{n+1}$!

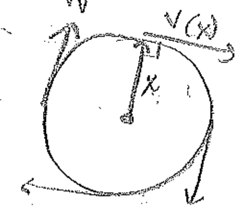
(g) If f has no fixed pts, then $\deg f = (-1)^{n+1}$, since f is he to $-\mathrm{Id}$: define a homotopy

$f_t(x) = \dfrac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$ (since $f(x) \to x$ does not pass through 0)

& $f_0 = f$, $f_1 = -x = -\mathrm{Id}$.

Ex: S^n has a continuous field of nonzero tangent vectors iff n is odd (Hairy ball thm).

Pf:



($\Leftarrow$): If $x = (x_1, x_2, \ldots, x_{2k-1}, x_{2k}) \in \mathbb{R}^{2k}$

$v(x) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2k}, -x_{2k-1})$ $n = 2k-1$

(swap 2 & take opp sign of 1 to get $\perp$)

($\Rightarrow$): Suppose S^n has a cts field of $\neq 0$ vectors.

wlog, $|v(x)| = 1$; then $\cos t\, \vec{x} + \sin t\, v(\vec{x})$, $t \in [0, \pi]$ lie in the unit circle in the plane spanned by $\{\vec{x}, v(\vec{x})\}$. So this is a homotopy from $\mathbb{1}_{S^n}$ to $-\mathbb{1}_{S^n}$

$\deg \mathbb{1}_{S^n} = 1$

$\deg -\mathbb{1}_{S^n} = (-1)^{n+1} = 1 \Rightarrow$ n odd

Ex: $\mathbb{Z}_2$ is the only nontrivial gp that acts freely on S^n if n is even. [recall: freely = w/o fixed pt], (action is by homeos.)

Pf: Note: The deg. of a homeo. is ± 1, since a homeo. is a h.e.

Let $G \curvearrowright S^n$. Then we get a homomorphism

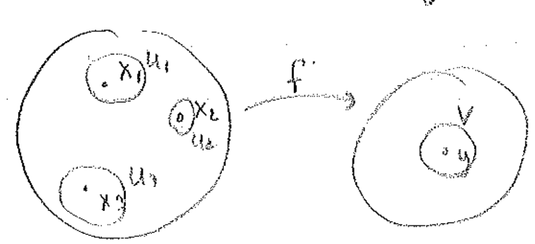
$\deg: G \to \mathbb{Z}$ Since G acts freely, the only elt that has a fixed pt is 1,

so if n even & $1 \neq g \in G$, $\deg g = (-1)^{n+1} = -1$

$\Rightarrow \ker(\deg) = 1 \Rightarrow G \cong \mathbb{Z}_2$. or G is trivial.

Computing Degree (when $f^{-1}(y) = \{x_1, \ldots, x_m\}$)

Let U_i be nbhds of x_i & they all map into V nbhd y, & U_i disj.



We have a comm. diagram:

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[d, phantom, "\cong" rotate=90] & \\
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "\text{inclusion}"] & H_n(V, V-y) \cong \mathbb{Z} \arrow[d, "\cong \text{ by excision}"] \\
H_n(S^n, S^n \setminus \{x_i\}) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"', "\text{proj}"] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[u, "q"] \arrow[r, "f_*"] \arrow[ul, "\cong"] & H_n(S^n) \arrow[u, "q", "\cong"']
\end{tikzcd}$$

from 2 ≅'s on left

from 2 ≅'s on right

by excision

here modding out by more

from LES b/c $H_n(S^n \setminus \{x_i\}) = 0$ ($\overset{\shortparallel}{A}$)

by ($A = S^n - y$ contractible)

from LES $\to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A)$

Def: We have $f_*: \underset{\mathbb{Z}}{\underset{\shortparallel}{H_n(U_i, U_i - x_i)}} \to \underset{\mathbb{Z}}{\underset{\shortparallel}{H_n(V, V-y)}}$

$$1 \longmapsto d_i$$

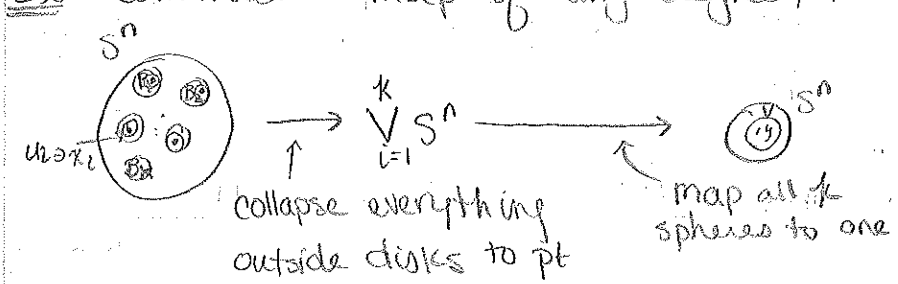
d_i is the local degree of f at x_i, denoted $\deg f|_{x_i}$.

Prop: $\deg f = \sum_{i=1}^{n} \deg f|_{x_i}$.

Pf: read.

Note: If f a homeo $U_i \to V$, $\deg f|_{x_i} = \pm 1$

Ex: Construct a map of any degree, k



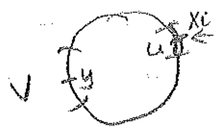
If this composition has local deg -1, then reflect in the wedge (so that it fixes base pt) & then deg is $+1$.

Ex: $f: S^1 \to S^1$, $f(z) = z^k$. $\Rightarrow \deg f = k$

Pf: $k = 0$, not surj, so $\deg f = 0 = k$ ✓

$k > 0$, then $f^{-1}(y) = \{x_1, \dots, x_k\}$, so we need

$\deg f|_{x_i} = 1 \ \forall\, 1 \le i \le k$.



← this is being stretched & rotated

- Can make it just a rotation via homotopy
- rotation is he to $\mathbb{1}_{S^1} \Rightarrow \deg = 1$.

Cellular Homology

X = CW complex

Lemma:

(a) $H_k(X^n, X^{n-1}) = \begin{cases} 0 & \text{if } k \neq n \\ \text{free abel. on basis of } n\text{-cells} & \text{if } k = n \end{cases}$

(b) $H_k(X^n) = 0$ if $k > 0$

(c) If $k < n$, $i: X^n \hookrightarrow X$ induces an isomorphism

$i_*: H_k(X^n) \to H_k(X)$

Pf: (a) (X^n, X^{n-1}) is a good pair, so $H_k(X^n, X^{n-1}) = \tilde{H}_k(X^n/X^{n-1})$

& X^n/X^{n-1} is a wedge of spheres.

(b) $\cdots \to H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \xrightarrow{i_*} H_k(X^n) \to H_k(X^n, X^{n-1}) \to \cdots$

If $k \neq n, n-1$, $H_{k+1}(X^n, X^{n-1}) = 0$ and $H_k(X^n, X^{n-1}) = 0$

then $H_k(X^{n-1}) \cong H_k(X^n)$

If $k > n$, then $H_k(X^n) \cong H_k(X^{n-1}) \cong \cdots \cong H_k(X^0) = 0$

↑ bunch of pts

(c) If $k < n$, then $H_k(X^n) \cong H_k(X^{n+1}) \cong \cdots \cong H_k(X^{n+m})$

If $\dim X < \infty$, done. ↑ maybe use (X^{n+1}, X^n) instead.

If $\dim X = \infty$, read.

all free abel. by (a) {

$$\begin{tikzcd}[column sep=small]
 & & & & H_n(X^{n+1},X^n)=0 & & \\
 & 0=H_n(X^{n-1}) \arrow[dr] & & H_n(X^{n+1}) \simeq H_n(X) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur, "i"] \arrow[dr, "j_n"] & & & & \\
 & H_{n+1}(X^{n+1},X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"'] & & H_n(X^n,X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1},X^{n-2}) \arrow[r] \arrow[ur] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0=H_{n-1}(X^{n-2}) \arrow[ur] & & &
\end{tikzcd}$$

Define $d_{n+1} := j_n \circ \partial_{n+1}$, $d_n := j_{n-1} \circ \partial_n$.

<u>Claim</u>: $\partial_n \circ \partial_{n+1} = 0$ [Then middle row is a chain complex!]

<u>Pf</u>: $\operatorname{Im} d_{n+1} = \operatorname{Ker} \partial_n \Rightarrow j_{n-1}$ goes to zero.

So, $H_n(X^n, X^{n-1})$ is free abel. w/ basis of n-cells of X^n

Thus we have a (cellular) chain complex. The homology is called <u>cellular homology</u>, denoted $H_n^{cw}(X)$.

<u>Thm</u>: $H_n^{cw}(X) \simeq H_n(X)$

<u>Pf</u>: $H_n(X) \simeq H_n(X^n)/\operatorname{Im} \partial_{n+1}$ b/c $\operatorname{Im} \partial_{n+1} = \operatorname{Ker} i$

Since j_n is injective, it maps $\operatorname{Im} \partial_{n+1}$ onto $\operatorname{Im} j_n \circ \partial_{n+1} = \operatorname{Im} d_{n+1}$

$\operatorname{Ker} \partial_n = \operatorname{Ker} d_n$, since j_{n-1} injective, &

$\operatorname{Ker} \partial_n = \operatorname{Im} j_n = H_n(X^n)$

So $H_n(X) \simeq \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$ □

<u>Ex</u> (1) $H_n(X) = 0$ if X has no n-cells

(2) If X has k n-cells, $H_n(X^n, X^{n-1}) = \mathbb{Z}^k$ &

$H_n(X) = {}^{\operatorname{Ker} d_n}/_{\operatorname{Im} d_{n+1}}$ & $\operatorname{Ker} d_n \leq \mathbb{Z}^k \Rightarrow$

$H_n(X)$ has at most k generators

(3) If X has no 2 of its cells in adjacent dimensions,

then $\to 0 \to \mathbb{Z}^{k} \to 0 \to \mathbb{Z}^{k-2} \to 0 \to \cdots$, so

$H_n(X)$ free on basis of n-cells.

Cellular boundary formula

$d_1: H_1(X^1, X^0) \to H_0(X^0)$ ← like simplicial homology

$n>1$: $d_n(e^n_\alpha) = \sum d_{\alpha\beta}\, e^{n-1}_\beta$ ← all (n-1)cells that e^n_α hits

↳ degree of map $S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$

∂e^n_α

collapse $X^{n-1} - e^{n-1}_\beta$, so $\partial e^{n-1}_\beta \to pt$, so get S^{n-1}_β

Ex: Σ_g : 1 0-cell
2g 1-cell
1 2-cell

dim: 3 2 1 0

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1=0} \mathbb{Z} \to 0$$

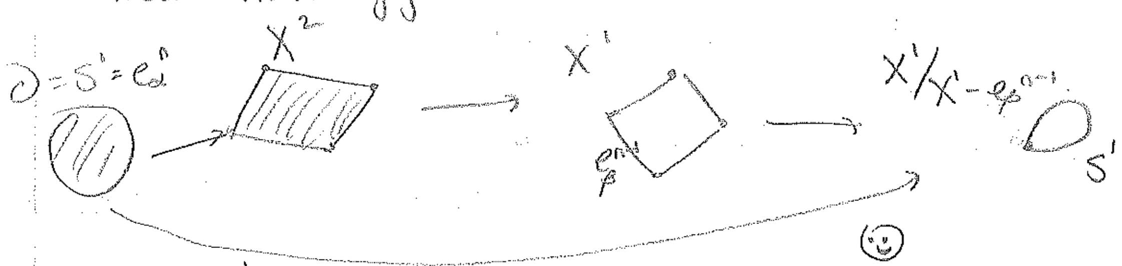
cellular chain complex

b/c know $H_0(\Sigma_g) = \mathbb{Z}$, ctd

d_2 : for each β, $\exists\, d_{\alpha\beta} = 1$ & $d_{\alpha\beta} = -1$, so everything cancels $\Rightarrow d_2 = 0$

Pick 3
- cov. sp.
- LES
- π_1
- cell. hom
- degree

12/4 Cellular Homology:



$\to C_{n+1}^{cw}(X) \xrightarrow{d_{n+1}} C_n^{cw}(X) \xrightarrow{d_n} C_{n-1}^{cw}(X) \to$

$d_n(e_\alpha^n) = \sum \deg ☺_i \, e_\beta^{n-1}$.

<u>Euler Characteristic</u> (X a CW complex)

$\chi(X) = \sum_n (-1)^n c_n$, $c_n = \#$ of n-cells

(in dim 2, this is V-E+F)

<u>Thm</u>: $\chi(X) = \sum (-1)^n \, rk(H_n(X))$.

($H_n(X) \cong \mathbb{Z}^r \oplus T$ by classification of f.g. abel gps, & $rank(H_n(X)) = r$)

<u>Fact</u>: If $0 \to A \to B \to C \to 0$ a SES of f.g. abel gps, then $rk(B) = rk(A) + rk(C)$.

<u>Pf</u> (of Thm): $0 \to C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} \dots$ a cellular chain complex

($c_k = rk(C_k)$ b/c C_k = free abel on k-cells)

$\left.\begin{array}{l} \ker d_n = Z_n = \text{cycles} \\ \operatorname{Im} d_{n+1} = B_n = \text{boundaries} \end{array}\right\} \subseteq C_n$

Have:

(*) $0 \to B_n \to Z_n \to H_n \to 0$ (i.e. $H_n = Z_n/B_n$)

(#) $0 \to Z_n \to C_n \to B_{n-1} \to 0$

Apply fact: $rk(Z_n) = rk(B_n) + rk(H_n)$

$rk(C_n) = rk(Z_n) + rk(B_{n-1})$

$\Rightarrow \sum_n (-1)^n rk(C_n) = \sum_n \big(\underbrace{\cancel{rk(B_n)} + \cancel{rk(B_{n-1})}}_{\text{cancel}} + rk(H_n)\big)(-1)^n$

$\Rightarrow \sum_n (-1)^n rk(C_n) = \sum_n (-1)^n rk(H_n)$.

Thus, the Euler characteristic is a top. invariant, & so does not depend on the CW-complex structure

Ex: $\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$.
- Since all orientable surfaces have diff χ (by above), they are all non-homeo.

$\chi(N_g) = 1 - g + 1 = 2 - g$. [recall $N_g = \langle a_1, \dots, a_g \mid a_1^2 \cdots a_g^2 = 1 \rangle$

Mayer-Vietoris Sequence (another LES)

Thm: If $X = A^\circ \cup B^\circ$, then $\exists$ LES

$$\cdots \to H_n(A\cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

Pf: We need a SES of chain complexes to get a LES

Def: $C_n(A+B)$ = sums of chains in A and in B $\subseteq C_n(X)$

i.e. $x+y$, where $x \in C_n(A)$ & $y \in C_n(B)$.

$\partial : C_n(A+B) \longrightarrow C_{n-1}(A+B)$

$\partial(x+y) = \partial x + \partial y$

$i: C_n(A+B) \longrightarrow C_n(X)$ induces an iso. on homology.
(from pf of excision)

SES:

in $C_n(A) \oplus C_n(B)$, you're ignoring that chains in A & B can interact in $A \cap B$.

$$0 \to C_n(A\cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$$
$$x \longmapsto (x, -x) \qquad (x,y) \longmapsto x+y$$

- $\ker \phi = 0$
- $\operatorname{Im} \phi \subseteq \ker \psi$ $\quad (x + -x = 0)$ i.e $\psi \circ \phi = 0$

  $\ker \psi \subseteq \operatorname{Im} \phi$ $\quad$ since $x+y=0 \Rightarrow \underset{\text{in } X}{x=-y} \Rightarrow x, y \in C_n(A\cap B)$
- ψ is surj

12/6 We can also augment the SES & get M-V seq. for reduced homology:

$$\begin{tikzcd}
0 \arrow[r] & C_0(A\cap B) \arrow[r, "\phi"] \arrow[d, "\varepsilon"] & C_0(A) \oplus C_0(B) \arrow[r, "\psi"] \arrow[d, "\varepsilon"] & C_0(A+B) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\phi"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\psi"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$$H_1(A\cap B) \xrightarrow{\phi} H_1(A)\oplus H_1(B) \xrightarrow{\psi} H_1(X) \to \tilde{H}_0(A\cap B)$$

* If $A\cap B$ is path-ctd, $\tilde{H}_0(A\cap B)=0$

Then $H_1(X) = H_1(A)\oplus H_1(B)/\operatorname{Im}\phi$. This is exactly the abelianized version of van Kampen:

$$\pi_1(X) \simeq \pi_1(A) * \pi_1(B) / \underbrace{\langle\langle i_{\alpha\beta}\, i_{\beta\alpha}^{-1} \rangle\rangle}_{\text{inclusions of } A\cap B \text{ into } A \,\&\, B}$$

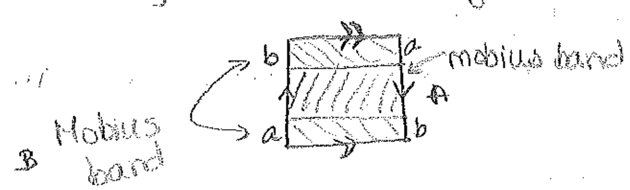
<u>Ex</u>: $X = S^n$ A, B the north & south hemispheres, so that $A\cap B = S^{n-1}$ (really make A & B a bit bigger $\Rightarrow$ so A as north hemi. is a def. retr. of a nbhd, & we should really use that nbhd).

$$\tilde{H}_i(S^{n-1}) \to \underset{\overset{\shortparallel}{0}}{\tilde{H}_i(A)} \oplus \underset{\overset{\shortparallel}{0}}{\tilde{H}_i(B)} \to \tilde{H}_i(S^n) \xrightarrow{\cong} \tilde{H}_{i-1}(S^{n-1}) \to 0$$

b/c A & B are contractible

By induction, starting w/ a pt, we get all homology gps.

<u>Ex</u>: $X = K$



$A\cap B = S^1$

$A \underset{he}{\simeq} S^1$, $B \underset{he}{\simeq} S^1$

Thus:

$$0 \to H_2(K) \xrightarrow{\phi} H_1(S^1) \hookrightarrow H_1(S^1)\oplus H_1(S^1) \to H_1(K) \to \overset{\overset{0}{\shortparallel}}{\tilde{H}_0(S^1)}$$

$1 \mapsto (2,-2)$

basis for $\mathbb{Z}\oplus\mathbb{Z}$: $(1,0)$ & $(1,-1)$,

so $H_1(K) \simeq \mathbb{Z}\oplus\mathbb{Z}_2$

$\Rightarrow \phi = 0$

$\Rightarrow H_2(K) = 0$

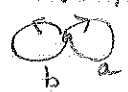
More cellular homology:

$$\cdots \to C^{cw}_{n+1}(X) \xrightarrow{d_{n+1}} C^{cw}_n(X) \xrightarrow{d_n} C^{cw}_{n-1}(X) \to \cdots$$

$H^{cw}_n(X) \cong H_n(X)$.

<u>Ex</u>: Find a space X whose $\tilde{H}_i(X) \cong \tilde{H}_i(*)$ but X is not contractible. Such spaces are called <u>acyclic</u>.

X is $S^1 \vee S^1$, then glue in 2 2-cells via

dim 2 

$a^5 b^{-3}$, $b^3 (ab)^{-2}$

$0 \to \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1 = 0} \mathbb{Z} \to 0$ since $H_0(X)$ must be $\mathbb{Z}$ (note this isn't reduced)

not nec. exact b/c these are chains

$d_2: \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$ det $= 1 \Rightarrow d_2$ is injective

1^{st} 2-cell corresp to $\binom{1}{0}$, 2^{nd} to $\binom{0}{1}$

d_2 is an $\cong$, since $\mathbb{Z}^2$ have same rank & det $= 1 \Rightarrow$ injective

$\Rightarrow$ all homology gps are 0.

Is X is contractible? No, b/c $\pi_1(X)$ is not trivial

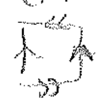
Pf: $\pi_1(X) \cong \langle a, b \mid a^5 b^{-3}, b^3 (ab)^{-2} \rangle \not\cong 1$ b/c can construct nontriv. hom to gp of rotations of a dodecahedron. This problem (triv or not?) is in general hard.

<u>Review</u>:

- Ch. 0 – notion of homotopy & constructing new spaces (cone, suspension, join...)
- Ch. 1 – fundamental gp $\to$ VK
  - covering spaces
    - lifting criterion $\begin{array}{ccc} & & \tilde{X} \\ & \overset{\tilde{f}}{\nearrow} & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$ Y p-ctd & lpctd & $f_*(\pi_1(Y)) \subseteq p_*(\pi_1(\tilde{X}))$
    - Galois correspondence: X very nice: p-ctd, loc p-c, SLSC

$\{$subgps of $\pi_1(X)\} / $conjugacy $\longleftrightarrow \{$covers of $X\} / $equiv of base pts

- Ch. 2 ~ Homology
  - $H_n^{\Delta}(X) \simeq H_n(X) \simeq H_n^{CW}(X)$

     ↑ mess 

  - Tools: - LES ⟨ (X,A), (Mayer-Vietoris), excision
    - $(\pi_1(X))^{Ab} = H_1(X)$ ...

- degree of maps $f: S^n \to S^n$
- Euler characteristic

12/11 ① $\tilde{H}_n(X) \simeq \tilde{H}_{n+1}(SX)$

$SX = CX/\{0\}\times X$, so $(CX, X\times\{0\})$

LES: $\tilde{H}_{n+1}(X) \to \underset{0}{\underset{\shortparallel}{\tilde{H}_{n+1}(CX)}} \to \tilde{H}_{n+1}(SX) \xrightarrow{\simeq} \tilde{H}_n(X) \to \underset{0}{\underset{\shortparallel}{\tilde{H}_n(CX)}} \to \cdots$

OR have $SX = A \cup B$, $A = CX$, $B = CX$, $A \cap B = X$

& by M-V, we have

$\underset{0}{\underset{\shortparallel}{H_{n+1}(A) \oplus H_{n+1}(B)}} \to H_{n+1}(SX) \xrightarrow{\simeq} H_n(X) \to \underset{0}{\underset{\shortparallel}{H_n(A) \oplus H_n(B)}} \to$

② X = p-ctd, loc. p-ctd, $\pi_1(X)$ finite

$Y = S^1 \times S^1$

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

need: $f_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R}^2)) = 1$

$\pi_1(Y) \simeq \mathbb{Z}^2$

③ Show $A \not\simeq B$

A:  B: 

Euler characteristic: doesn't work

$\pi_1(A) = \langle a \rangle / \langle\langle a^4 \rangle\rangle$ (Check vK for CW-complexes)

$\pi_1(B) = \langle b \rangle / \langle\langle b^6 \rangle\rangle$

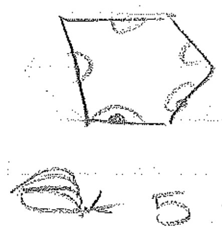
OR homology ($H_1 = \pi_1$ in this case) ex: (from last year's final)

$\pi_1 = \langle a \mid a^2 a^{-1} \rangle$
$= \langle a \mid a \rangle$
$= 1$

$\pi_1 = \langle b \mid b^2 b^{-3} \rangle$
$= \langle b \mid b \rangle$
$= 1$

u

3 things glued together



5 things glued together

Can use Local Homology:

$H_n(U, U - \{x\}) \cong H_n(V, V - \{f(x)\})$

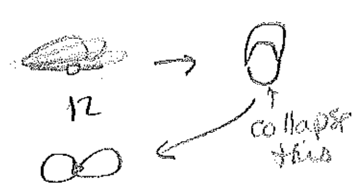
use a LES

$f: U \to V$ a homeo.

$H_n(U - \{x\}) \to H_n(U) \to H_n(U, U - \{x\}) \to H_{n-1}(U - \{x\}) \to$

$H_n(U) = 0$

b/c is contractible



for V, either V is in interior & $V - \{x\}$ homog. of ⊙
or V on edge, & $V - \{x\}$ is wedge of 4 circles

OR: try w/ pt set topology...

④ Calculate homology $S^{2n} \times S^{4n}$, $n \geq 0$.

If $n > 0$ $e^0 \cup e^{2n} \times e^0 \cup e^{4n} = e^0 \cup e^{4n} \cup e^{2n} \cup e^{6n}$

if $n \geq 1$, no 2 cells are in adj. dim

$0 \to \underset{6n}{\mathbb{Z}} \to 0 \to \mathbb{Z} \to \underset{4n}{\mathbb{Z}} \to 0 \to \underset{2n}{\mathbb{Z}} \to \underset{0}{\mathbb{Z}} \Rightarrow H_k = \begin{cases} \mathbb{Z} & k = 6n, 4n, 2n \\ 0 & \text{else} \end{cases}$

If $n=0$, $S^0 \times S^0$, $S^0 = \{x,y\}$ $=$ $\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix}$

$H_0(S^0 \times S^0) = \mathbb{Z}^4$, $H_k = 0$ else.