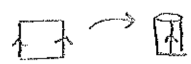
9/4

Hatcher, Topology
Ch. 0-2
Top. qual slightly different
next summer.

Math 751
dymarz@math.wisc.edu
VV 509
math.wisc.edu/~dymarz/751

Midterm 1: 30% Takehome (Oct.)
Midterm 2: 40% In Class (Last wk of class)
HW: 30% About every 2 wks - no late HW (1 exception)

(X, τ) Top. sp.
set ↳ open sets

Review:
- subsp. topology $A \subseteq X$
- product top. $X \times Y$
- quotient top. $X/\sim$ Ex: 
- continuous maps
- homeomorphisms $f: X \to Y$

Standing Assumption: • All maps are cts. (or cnts)
• All spaces are "nice."

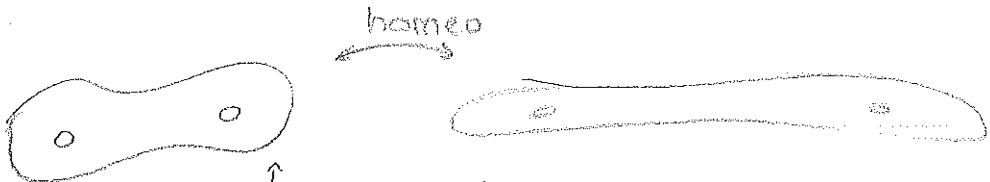
Q: Which top sp's are equivalent? (ie. homeomorphic)
(Hard)

Goal (of alg. top)
• Attach alg. invariants to top. sp's (i.e number, group, & fundamental gp, homology gp)

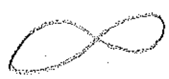
Other uses:
• Get info about gp theory using top. sp's attached to them — top. proof that subgp of free gp is free, using covering sp. theory.
• analysis, calculus, diff. geom, etc.

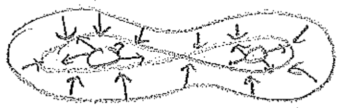
<u>Outline</u>:

- Ch. 0: - relaxes notion of homeomorphism to weaker notion of <u>homotopy</u>.
  - examples / constructions
- Ch. 1: - fundamental gp
  - Van Kampen Thm
  - Covering sp's ..
- Ch. 2: Homology



not homeo, but similar





can continuously squish the 1^{st} into the 2^{nd}. But inverse map not cts.

<u>Def</u>: $A \subseteq X$ subsp. A <u>deformation retraction</u> of X onto A is a family of maps $f_t : X \to X$, $t \in I = [0,1]$ satisfying:

- $f_0 : X \to X$ is the identity
- $f_1 : X \to A$, i.e. $f_1(X) = A$
- $f_t|_A$ is the id_A $\forall t$
- $F : X \times I \to X$ is cts.

<u>Ex</u>: Cylinder & möbius band both deformation retract to a circle.

<u>Ex</u>: $\phi : X \to Y$. The mapping cylinder $M_\phi = X \times I \sqcup Y / \sim$

$(x, 1) \sim f(x)$
$\quad \in X \times I \qquad \in Y$

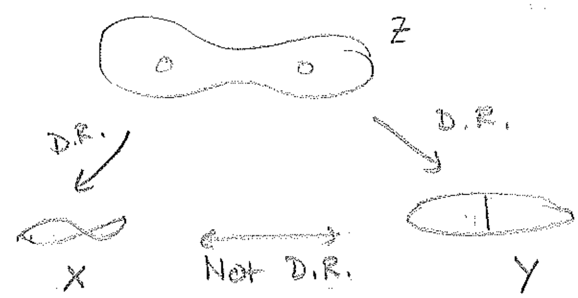
$\} M_\phi$

Claim: M_ϕ deformation retracts to Y.

$F: M_\phi \times I \to M_\phi$

ex:

Note:
Deformation retraction is not an equivalence relation. fails transitivity.

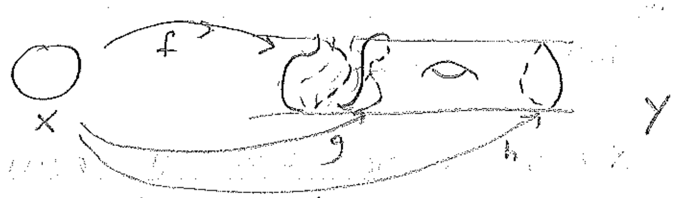


BUT: define an equivalence $X \sim Y$ if both are def. retractions of some bigger space Z. This equivalence is equivalent to another equivalence, homotopy equivalence.

9/6 Homotopy Equivalence

Def: Two maps $f, g: X \to Y$ are homotopic if $\exists$ $F: I \times X \to Y$ (cts) w/ $F(0,x) = f(x)$ & $F(1,x) = g(x)$. Think of it as a family of maps f_t w/ $f_0 = f$, $f_1 = g$.

ex

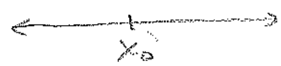


f & g are homotopic
g & h are not homotopic, b/c the hole gets in the way

We write $f \underset{h.e.}{\simeq} g$

ex: Check this is an equiv. relation

The deformation retraction $F: X \times I \to X$ has
$f_0 = Id_X$, $f_1 = r: X \to X$ (w/ $r(A) = A$) r is called a <u>retraction</u>
(ie a top. proj onto a subsp — $r^2 = r$)
so F is a homotopy equiv. btwn Id_X & r, i.e.
Id_X & r are homotopic.

<u>Ex</u>: 

$f_t(x) = (1-t)x + t x_0$
$f_0(x) = x$
$f_1(x) = x_0$

$F: \mathbb{R} \times I \to \mathbb{R}$
$(x,t) \mapsto f_t(x)$

so a line & a pt are homotopic.
(true for $\mathbb{R}^n$, as well → so $Id_{\mathbb{R}^n}$ is homotopic to the constant map)
* The identity map on $\mathbb{R}^n$ is null homotopic, i.e.
<u>Def</u>: A map $f: X \to Y$ that is h.e. to a constant map is <u>null homotopic</u>.

<u>Ex</u>: 

Can map $f: X \to x_0 \in X$, a retraction,
but there is no deformation retraction.

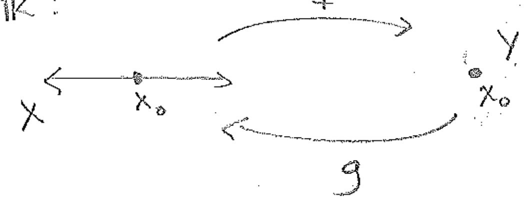
<u>Ex</u>: Any space that's not path-ctd will have a retraction that's not a def retr. B/c, a def retr. creates paths, just by following a pt x_0 through $f_t(x_0)$.

<u>Def</u>: Two spaces X & Y are <u>homotopically equivalent</u> if $\exists$ maps $f: X \to Y$, $g: Y \to X$ s.t. $f \circ g \underset{he}{\simeq} Id_Y$ & $g \circ f \underset{he}{\simeq} Id_X$.

<u>ex</u>: Check this is an equiv. rel.

Ex: $\mathbb{R}^n \underset{h.e.}{\simeq} \{x_0\}$ but they are not homeomorphic.

In $\mathbb{R}$:



$f \circ g : Y \to Y$ is the identity on Y.

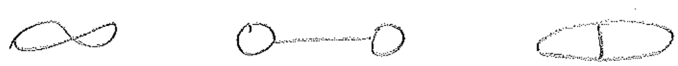
$g \circ f : X \to X$. $g \circ f(x) = x_0$. We saw above that this is h.e. to Id_X.

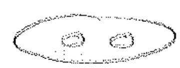
Def: If X is h.e. to a pt, we say X is <u>contractible</u>.

Ex: $A \subseteq X$. If A is a def. retr. of X, then $X \underset{he}{\simeq} A$.

$i : A \to X$ $i \circ r : X \to X$ $i \circ r \underset{he}{\simeq} Id_X$ by def. retr.

$r : X \to A$ $r \circ i : A \to A$, $r \circ i = Id_A$

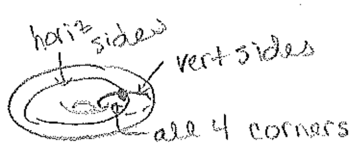
Ex:  all h.e.

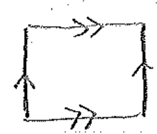
(all def. retr's of )

HW due Fri 9/20

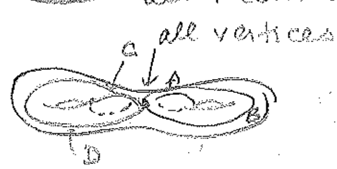
Spaces

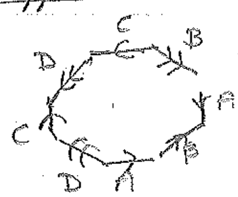
Torus: T^2 (genus 1)





Σ_2



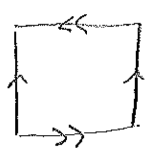


Σ_n

4n-gon

(surface of genus n)

Klein Bottle



So surfaces can be thought of as taking a pt, gluing to it $4g$ D^1's & then gluing to that one D^2.

↑ 1-dim'l disc

Cell-Complexes aka CW-complexes.

X^0 : discrete set of pts

X^1 : need: $\{D^1_\alpha \mid \alpha \in A\}$ D^1 ∂D^1

$\{\Phi_\alpha : \partial D^1_\alpha \to X^0\}$

so $X^1 = X^0 \sqcup_\alpha D^1_\alpha /_\sim$: for $x \in \partial D_\alpha$, $x \sim \Phi_\alpha(x)$

$\vdots$

$X^n = X^{n-1} \sqcup_\alpha D^n_\alpha / \sim \quad x \sim \phi_\alpha(x)$. [$\Phi_\alpha$ are "attaching maps"]

9/9 $\partial D^1 = S^0$, $\partial D^2 = S^1$, $\partial D^3 = S^2$

Let $e^n_\alpha = D^n_\alpha \setminus \partial D^n_\alpha$ (the open n-cell). Then
$X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$.

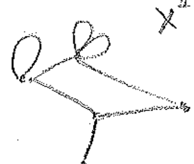
Notes:

- If $X = X^n$ for some n (i.e. procedure stops), X is an n-dim'l CW-complex.
- If $X = \bigcup_n X^n$ with weak topology [A is open (cl.) if $A \cap X^n$ is open (cl.) $\forall n$]
- X^k is the k-skeleton of X.

Ex:

$T^2 = \underbrace{e^0 \cup e^1 \cup e^1}_{X^1 = \text{ 1-skeleton}} \cup e^2$

Ex: $X = X^1$



a graph
(only objects are pts & D^1's)

Ex: $S^n = \underset{(1\text{ pt})}{e^0} \cup e^n \qquad X^0 = \{*\}$

D^n

$\Phi_\alpha(x) \longrightarrow *$ $\Rightarrow$

$\to$ glue ∂D^n to a pt.

OR: $e^0 \cup e^1$ S^{n-1}

glue one disk in front & one in back

$S^2 = e^0 \cup e^1 \cup 2e^2$

OR

$S^2 = 2e^0 \cup 2e^1 \cup 2e^2$

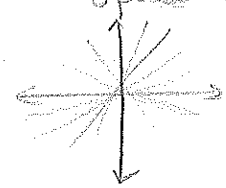
2 pts $\cup$ 2 segs = circle, then glue 2 disks one in front, one in back

$\vdots$

$S^n = 2e^0 \cup 2e^1 \cup \cdots \cup 2e^n$

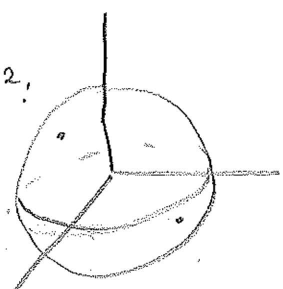
Ex: $\mathbb{R}P^n$ - space of lines through 0 in $\mathbb{R}^{n+1}$.

$\mathbb{R}P^2$:



or: unit vectors w/ $\vec{v} \sim -\vec{v}$, so it's the quotient of an n-sphere $S^n/\sim$.

$\mathbb{R}P^2$:



In $S^2 \setminus S^1$ (equator), identifying bottom cap w/ top cap so after identification, there's one 2-cell.
On S^1, id. opp pts, as in $\mathbb{R}P^1$

So, $\mathbb{R}P^2 = \mathbb{R}P^1 \cup e^2$ & $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup e^n$

Thus $\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$

$\mathbb{R}P^\infty = \bigcup_n \mathbb{R}P^n = e^0 \cup \cdots \cup e^n \cup \cdots$

Ex: $\mathbb{C}P^n$ - space of complex lines through origin in $\mathbb{C}^{n+1}$
- read in book.

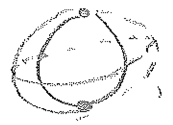
Def:

$\Phi_\alpha : D_\alpha^n \to X$ characteristic map

$\Phi_\alpha : D_\alpha^n \xrightarrow{\Phi_\alpha} X^n \hookrightarrow X$

first map into n-skeleton.

Def: A subcomplex $A \subseteq X$ is a closed subspace that is a union of cells. Depends on the CW decomp of X.

Ex: If S^2  $S^1 \subset S^2$ a subcomplex if $S^2 = e^0 \cup e^1 \cup 2e^2$

If $S^2 = e^0 \cup e^2$, S^1 not a subcomplex of S^2.

Warning: It does not work to take the closure of cells:

Ex: $X =$  $A = e^2$

2-cell

1-skeleton

$X = e^0 \cup e^1 \cup e^2$

$\bar{A}$ is not a union of cells

$\bar{A} =$  e^2 w/ seg.

Def: We call (X, A), A a subcomplex, a CW-pair.

Operations on Spaces

- Products: X, Y, get $X \times Y$. If X, Y are CW-complexes, so is $X \times Y$. The cells of $X \times Y$ are $e_\alpha^n \times e_\beta^m$ for $e_\alpha^n \subseteq X$ & $e_\beta^m \subseteq Y$.

ex: $\underbrace{S^1}_{e_\alpha^0 \cup e_\alpha^1} \times \underbrace{S^1}_{e_\beta^0 \cup e_\beta^1} = T^2$

Have: $e^0_{\alpha\times\beta} = e_\alpha^0 \times e_\beta^0$ } 0 cell

$$\left.\begin{array}{l} e_\alpha^0 \times e_\beta^1 \\ e_\alpha^1 \times e_\beta^0 \end{array}\right\} \text{1 cells}$$

$$\left. e_\alpha^1 \times e_\beta^1 \right\} \text{2 cell}$$

Warning: If X & Y are not f.d., then the usual product topology is coarser than the CW-complex topology.

ex. What do the attaching maps look like? (products of gluing maps)

- Quotients

(X,A) a CW-pair, $X \to X/A$.

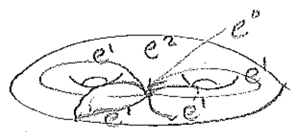
X/A is a CW-complex

X:  ← A = half sphere = $e^0 \cup e^1 \cup e^2$

$X \setminus A$ is not a subcomplex, but when you collapse A to a pt, throw in a pt.

X/A has cells $X \setminus A \cup e^0$.

9/11 Ex:



(X, X') a CW-pair

$X/X' = e^2 \cup e^0 = S^2$, b/c only one way to attach a disk to a pt.

- attaching maps will factor through quotients.

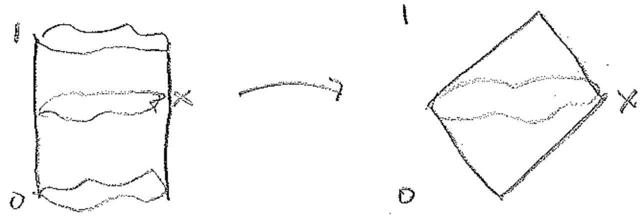
- Suspension

Given X, define the suspension of X,

$SX = X \times I/\sim$

$X \times \{0\} \sim \{*\}$ $\quad ((x_1,0) \sim (x_2,0) \ \forall\, x_1, x_2 \in X$

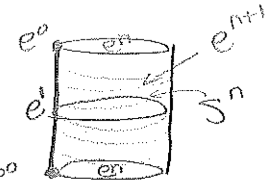
$X \times \{1\} \sim \{*\}$ $\quad (x_1,1) \sim (x_2,1) \ \forall\, x_1, x_2 \in X\ )$



ex Define a cone on X.

- If X is a CW-complex, then $X \times I$ is a CW, so is $\sim$ a subcomplex? Yes, b/c just copies of $X \times \{1\}$ or $X \times \{0\}$. Then SX is a CW-complex.

Ex: SS^n

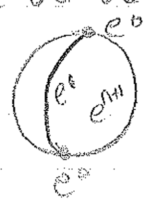


$S^n \times I$
$e^0, e^n \quad e^0, e^0, e^1$

$S^n \times I = \cancel{e^0} \cup \cancel{e^0} \cup e^1 \cup e^n \cup \cancel{e^n} \cup e^{n+1}$

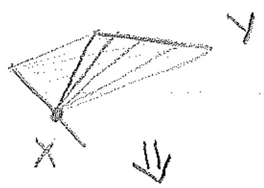
$e^0 \cup e^0$

$SS^n = e^0 \cup e^0 \cup e^1 \cup e^{n+1} = S^{n+1}$



- Join

Given X, Y, the join is $X * Y$.



- every pt in X is ctd to every pt in Y by a ~~seg~~.

$\Downarrow$

$X * Y =$ 

$X * Y = X \times Y \times I / \sim$

$(x, y_1, 0) \sim (x, y_2, 0)$, ie only copy of Y at $t = 0$.
$(x_1, y, 1) \sim (x_2, y, 1)$ —— " —— X at $t = 1$

If X, Y are CW-complexes, so is $X * Y$ (b/c product is & equivalence is subcomplex)

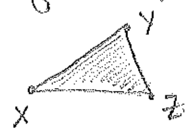
We can also write (x, y, t) as
$t_1 x + t_2 y$ (a formal lin. comb.), $0 \le t_i \le 1$,
$t_1 + t_2 = 1$.

So our equivalence, can write $(1-t)x + ty$.

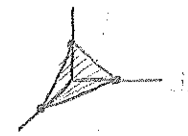
Can extend def to $X_1 * \cdots * X_n$. Think of it as formal lin combs $t_1 x_1 + \cdots + t_n x_n$, $\sum t_i = 1$, $0 \le t_i \le 1$.

Ex: (1) $\{*\} * \{*\} =$

(2) Join of 3 pts?



- if 3 pts are unit vectors in $\mathbb{R}^3$, then $t_1 x + t_2 y + t_3 z = 1$, so get



(3) Join of 4 pts? tetrahedron

⋮

(4) Join of n pts? an $(n-1)$-simplex

- Wedge Sum

Given X, Y & $x_0 \in X$, $y_0 \in Y$, then

$X \vee Y = X \sqcup Y / x_0 \sim y_0$

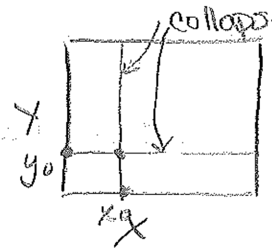
Ex:

- If X, Y are CW-complexes, so is $X \vee Y$.

- Smash Product

Given X, Y, then $X \wedge Y = X \times Y / X \vee Y$

collapse these: $X \times \{y_0\} \cup \{x_0\} \times Y$



- $X \wedge Y$ has a CW structure if X, Y are, if you pick x_0 & y_0 to be zero cells in X, Y.

Ex: $S^m \wedge S^n$, $S^m \times S^n = \underbrace{e^0 \cup e^n \cup e^m}_{\text{copy of } S^m \text{ \& } S^n \text{ joined at a pt}} \cup e^{m+n}$

$(e^0 \cup e^m) \wedge (e^0 \cup e^n)$

$\downarrow e^0$

$\Rightarrow S^m \wedge S^n = e^0 \cup e^{m+n} = S^{n+m}$

Ex: $S^1 \wedge S^1 = S^2$

quotient out by 1-skeleton, as before.

9/13 **Back to Homotopy Equivalence**

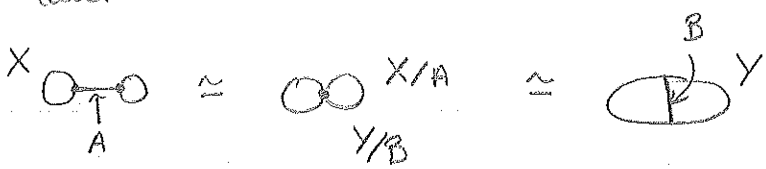
Recall, $\exists\ f: X \to Y$ & $g: Y \to X$ s.t. $f \circ g \underset{h.e.}{\simeq} \mathbb{1}_Y$ & $g \circ f \underset{h.e.}{\simeq} \mathbb{1}_X$.

Prop: If (X, A) is a CW-pair, and A is contractible, then $X \xrightarrow{f} X/A$ is a homotopy equivalence (i.e., there is an "inverse" g s.t. both compositions are h.e. to $\mathbb{1}$).

OR: $X \underset{h.e.}{\simeq} X/A$.

(Proof later)

Ex:



$X = e^0 \cup e^0 \cup e^1 \cup e^1 \cup e^1$

Ex: G is a ctd graph w/ fin. many vertices & edges

Claim: $G \simeq \vee_n S^1$ (wedge of n circles)

maximal tree = A

- Contract the maximal tree (using the greedy algorithm)

ex: A tree is contractible

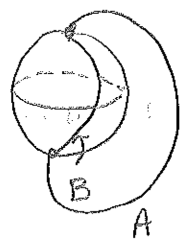
Question: $\vee_n S^1 \underset{h.e.}{\simeq} \vee_m S^1$? No.

Note: graphs have different Euler characteristics ($v - e + f$) [no faces in graphs]

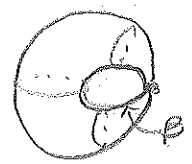
$\vee_n S^1$: $1-n$ vs. $\vee_m S^1$: $1-m$

<u>Side note</u>: Can define E.C. for cw complexes:
$\chi = k_0 - k_1 + k_2 - k_3 + \cdots$, where $k_i = \#$ of e^i cells.
[can also be defined for general X using homology]
<u>Fact</u>: Euler char. is a homotopy invariant, i.e. if X & Y have different χ's, then $X \not\simeq_{he} Y$.
Pf: later?

<u>Ex</u>: X



$\longrightarrow$ X/A

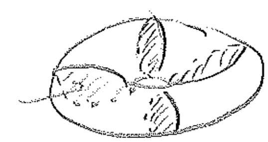


$\searrow$ X/B :

$= S^2 \vee S^1$

$X/A \simeq_{h.e.} X/B$
(Since both h.e. to X)

<u>Ex</u>:



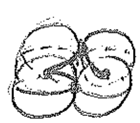
Torus w/ n disks glued inside
Each disk is contractible.

$\leftarrow$ one piece is a sphere & after contract disks still a sphere

$\downarrow$

each piece ctd by a pt.

Get <u>necklace</u>:



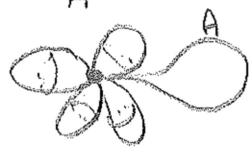
X/A & Y/A

$\simeq_{h.e.}$

Y

$\downarrow$ Y/B

Collapse B:



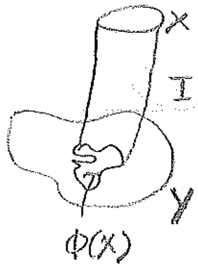
$= (\vee_n S^2) \vee S^1$

Attaching Spaces

$A \subseteq X_1 \qquad f : A \to X_0$

$X_0 \sqcup_f X_1 = X_0 \sqcup X_1 / a \sim f(a) \ \forall a \in A$ (glue X_0 to X_1 along the image of A)

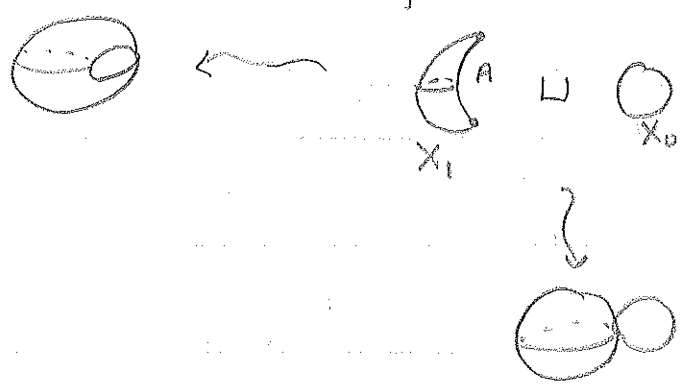
Ex: Mapping cylinder of ϕ



$(X \times I) \sqcup_f Y$, $A = X \times \{1\}$
$f(x,1) = \phi(x)$.

Thm: If (X_1, A) is a CW pair and $f, g : A \to X_0$, then $X_0 \sqcup_f X_1 \underset{he}{\simeq} X_0 \sqcup_g X_1$ if $f \underset{he}{\simeq} g$.

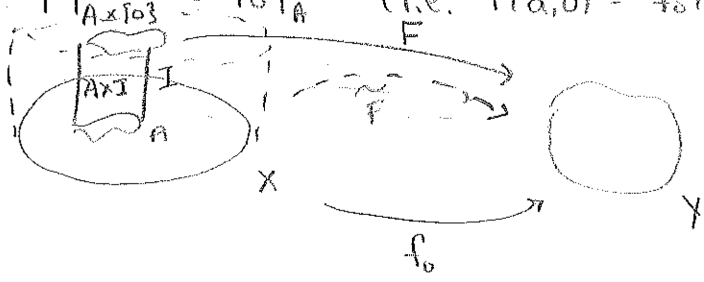
9/16 Ex: All examples from last time can be thought of this way → collapsing subsp's



via wrap A around
or via pt

Homotopy Extension Property (HEP)

Idea: Have $f_0 : X \to Y$ & have homotopy $F : A \times I \to Y$, $A \subseteq X$, s.t. $F|_{A \times \{0\}} = f_0|_A$ (i.e. $F(a,0) = f_0(a)$)



Goal:
$\tilde{F} : X \times I \to Y$ s.t.
$\tilde{F}|_{A \times I} = F$ &
$\tilde{F}|_{X \times \{0\}} = f_0$.

Def: A pair (X, A) [not nec. cw] has HEP if for every problem as above, there is a sol'n, i.e., $\forall Y, f_0, F$, $\exists$ an $\tilde{F}$.

Prop: (X, A) has HEP iff $X\times\{0\} \cup A\times I$ is a retract of $X \times I$.

($r: X\times I \to X\times\{0\}\cup A\times I$ fixing subsp. plus)

Pf: ($\Rightarrow$): Let $Y = X\times\{0\}\cup A\times I$

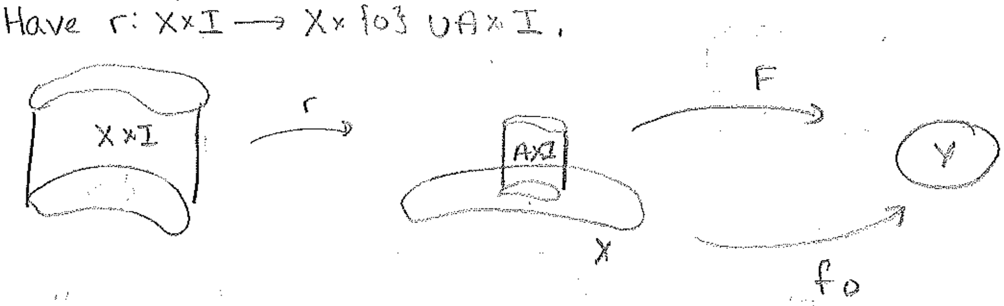
$f_0 : X \hookrightarrow X\times\{0\}\cup A\times I$ an embedding

$F: A\times I \hookrightarrow X\times\{0\}\cup A\times I$ an inclusion

(they agree on sp. b/c inclusion)

By HEP, $\exists\, \tilde{F}: X\times I \to X\times\{0\}\cup A\times I$. Since $\tilde{F}$ extends f_0 & F, which both are inclusions & so fix subsp, this is a retraction.

($\Leftarrow$): Have $r: X\times I \to X\times\{0\}\cup A\times I$.



For any Y and $F: X\times I \to Y$ and $f_0 : X \to Y$ call $g: \underbrace{X\times\{0\}}_{f_0} \cup \underbrace{A\times I}_{F} \to Y$. Then $\tilde{F} = g\circ r$.

* g might not be cts if A is not closed, so this proof only works if A closed OR if X is hausdorff.

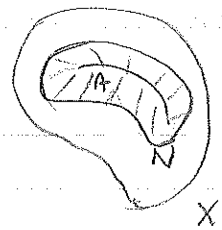
ex: Why true if X hausdorff

ex: If $X\times\{0\}\cup A\times I$ is a retract of $X\times I$, & X is Hausdorff, then A is closed.

Ex: It is not true that $\forall$ pairs (X,A) $\exists$ retr.
$r: X\times I \to X\times\{0\}\cup A\times I$:

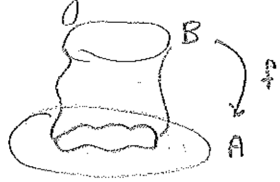
ex $\quad X = I$, $A = \{1, 0, 1/2, 1/4, \dots\}$ problem: "bunching around 0"

Ex: (X,A) has HEP if A "has a mapping cylinder neighborhood," ie



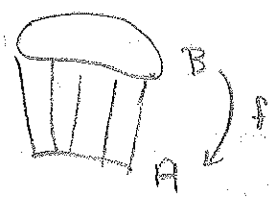
N = cl. nbhd of A, $B = \partial N$ (anything s.t. $N\setminus B$ open) for ex.
$N\setminus B$ = open nbhd of A
$f: B\to A$
$h: M_f \to N$ a homeo
M_f = mapping cylinder of f
an $h|_{A\cup B} = \mathrm{Id}$

[M cylinder:



$B\times I \sqcup f/\sim$,
$(b,1)\sim f(b)$]

In Ex,

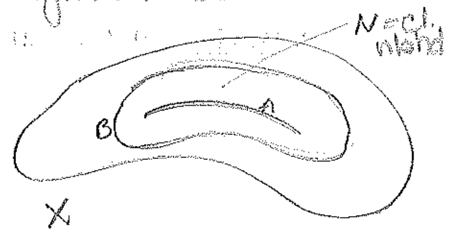
M_f 

- If glue M_f into X, then M_f is N.

Picture idea of why: (next time)

9/18 No class Fri. 9/27.

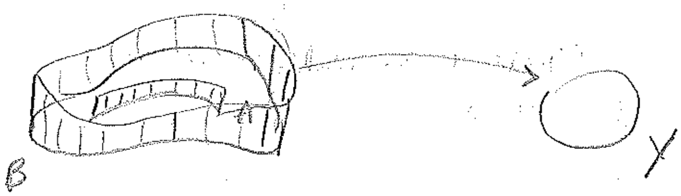
Prop: (X,A) has HEP if A has a "mapping cylinder neighborhood."



$N \underset{h}{\overset{\text{homeo}}{\cong}} M_\phi$ ← mapping cylinder, where $\phi: B \to A$, w/ B is any cl. set s.t. $N \setminus B$ is an open nbhd of A (such as $B = \partial N$)

→ $h: N \to M_\phi$ where $h|_{A \cup B} = Id.$

Pf: ① Claim: $(N, A\cup B)$ has HEP.



⇒ given homotopy $F: A\cup B \to Y$ can extend it to one from $N \to Y$ (i.e., can fill in)

② Given $F: A\times I \to Y$ & $f_0: X \to Y$. On $X\setminus(N\setminus B)$ do constant homotopy, ie. $\tilde{F}(x,t) = f_0(x)$ on $\underbrace{X\setminus(N\setminus B)}_{X\setminus N \text{ and } B}$.

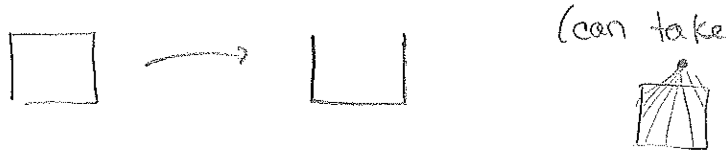
This defines $\hat{F}: B\times I \to Y$ (f_0), so restrict everything to N & use claim 1, ie I have an HEP problem for $(N, A\cup B)$, use ① to solve it. Then define $\tilde{F}: X\times I \to Y$ as f_0 on $X\setminus(N\setminus B)$ & as sol'n to ① on N.

ex: Write down $\tilde{F}$ explicitly & show its cts.

Pf of ①:

$r: I\times I \longrightarrow I\times\{0\} \cup \partial I \times I$, retraction induces

(can take pt & use radial proj)



$\tilde{r}: B\times I\times I \to B\times I\times\{0\} \cup B\times\partial I\times I$ a retraction (id on B, r on $I\times I$ → product map)

$M_\phi = B\times I \sqcup A/\sim$, $(b,1)\sim\phi(b)$, the mapping cyl induces a retraction

$\hat{r}: M_\phi \times I \to M_\phi\times\{0\} \cup (A\cup B)\times I$

by prop, since have retraction onto $A \cup B$, so $(M_\varphi, A\cup B)$ has HEP, & $M_\varphi \cong N$ via a homeo that fixes A & B, so $(N, A\cup B)$ has HEP. □

<u>Prop</u>: If (X,A) is a CW-pair, then $X\times\{0\} \cup A\times I$ is a deformation retraction of $X\times I$.

<u>Cor</u>: (X,A) has HEP.
<u>Pf</u>: A def retr. is a retraction.

<u>Pf of Prop</u>:

- <u>A single closed cell</u>: There is a def. retraction
  $r: D^n\times I \to D^n\times\{0\} \cup \partial D^n \times I$

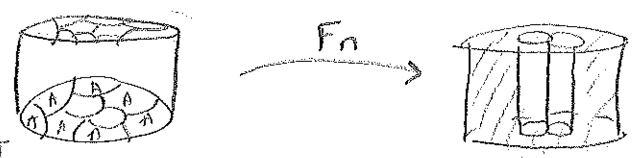
$D^2\times I$ $\qquad$ $D^2\times\{0\}\cup\partial D^2\times I$ $\qquad$ (proj. again)

<u>ex</u>: Write down this retraction in $\mathbb{R}^3$, using pt $(0,0,2)$, for ex. (in gen. embed in $\mathbb{R}^{n+1}$)

To see this is a def. retr, define
$r_t = (1-t)\cdot Id + t\cdot r$ $\qquad$ [check it's cts]

- <u>n-skeleton</u>: The above def. retr. gives rise to a def retr $X^n\times I \to X^n\times\{0\} \cup (X^{n-1}\cup A^n)\times I$.

9/20



"punch down", as above, for all cells not in A.

$X^n\times I$

$X^n\times\{0\} \cup (A^n\times I)\cup(X^{n-1}\times I)$

↑ boundary of the $D^n\times I$, but that's glued into the X^{n-1} skeleton.

[details: this is all happening on quotient, so really doing 1st part on whole space & the quotienting by attaching maps]

$F_n : (X^n \times I) \times I \to X^n \times I$, the def. retr.

- To get $F : (X \times I) \times I \to X \times I$ (s.t. $F_1(X \times I) = X \times \{0\} \cup A \times I$) need to do this for all n-skeletons. Do F_n on $[1/2^{n+1}, 1/2^n]$. Note: nothing happens on n-skeleton from $[0, 1/2^{n+1}]$, so map F is cts. (we only need to check continuity on n-skeleton $\forall n$).
  - 0 is the only issue, if you're ∞ dim'l, but b/c of the way we def. the top, if it works for the k-skeleton $\forall k$, then works glued together.

<u>Prop</u>: If (X,A) has HEP, and A is contractible, then $q : X \to X/A$ is a homotopy equivalence.

<u>Pf</u>: [NTS: $\exists g : X/A \to X$ s.t. $q \circ g \simeq Id_{X/A}$ & $g \circ q \simeq Id_X$]

Have homotopy from Id_A to pt map b/c A contractible: $f_t : A \to A$, $t \in [0,1]$ & $f_0 = \mathbb{1}_A$ & $f_1(A) = a_0$, a_0 fixed.

Extend $f_0 = \underline{Id} : X \to X$ & $f_t : A \to A$. Call this ext'n $f_t : X \to X$.

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f_t}"'] & X/A \end{tikzcd}$$

($f_t(A) \subset A$, so factors through quotients)

$$\overline{f_t}(\bar{x}) = \overline{f_t(x)} \qquad (q(x) = \bar{x})$$

At $t = 1$: $A \longrightarrow a_0$

$$\begin{tikzcd} x \in A \arrow[r, "f_1"] \arrow[d, "q"'] & a_0 \\ \bar{a} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f_1}"'] \arrow[ur, dashed, "\exists g"] & X/A \end{tikzcd}$$

Define g s.t. $g \circ q = f_1$, so $g(\bar{a}) = a_0$, $g(\bar{x}) = f_1(x)$

Claim: $q \circ g = \bar{f}_1$

Pf: $q \circ g(\bar{x}) = q \circ g \circ q(x) = q \circ f_1(x) = \bar{f}_1 \circ q(x) = \bar{f}_1(\bar{x})$

Finally, show $g \circ q \underset{he}{\simeq} \mathbb{1}_X$ & $q \circ g \underset{he}{\simeq} \mathbb{1}_{X/A}$:

$g \circ q \underset{he}{\simeq} f_1 \underset{he}{\simeq} f_0 = \mathbb{1}_X$ & $q \circ g \underset{he}{\simeq} \bar{f}_1 \underset{he}{\simeq} \bar{f}_0 = \mathbb{1}_{X/A}$

via f_t via $\bar{f}_t$, also a homotopy

∎

Prop: (X_1, A) a CW-pair, $f, g : A \to X_0$, & $f \underset{he}{\simeq} g$, then $X_0 \sqcup_f X_1 \underset{he}{\simeq} X_0 \sqcup_g X_1$ rel X_0.

fix

$X \xrightarrow{\phi} X$ & $X \xrightarrow{\psi} X$

Def: If $\exists$ a homotopy equivalence in both directions that fix X_0 ptws, then X & Y are homotopic rel X_0.

X_0 ← a subsp   & $\phi \circ \psi$ & $\psi \circ \phi$ fix X_0 ptws

Pf of Prop: $F : A \times I \to X_0$ $(f \simeq g)$ use HEP to extend F to $X_1 \times I$

$X_0 \sqcup_F (X_1 \times I)$

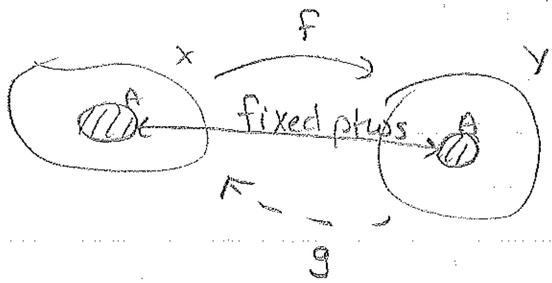
[define bigger sp that has both as def. retr.]

- this contains $X_0 \sqcup_f X_1$ at $t=0$ & $X_0 \sqcup_g X_1$ at $t=1$
- we have a deformation retraction $X_0 \sqcup_F (X_1 \times I) \to X_0 \sqcup_f X_1$, induced by $X_1 \times I \to X_1 \times \{0\} \cup \underbrace{A \times I}$ (from CW pair) glued to X_0, so this def. retr. applied to $X_0 \sqcup_F (X_1 \times I)$ yields only X_1 (b/c $f : A \to X_0$)

we also have a def. retr. $X_0 \sqcup_F (X_1 \times I) \to X_0 \sqcup_g X_1$ induced by $X_1 \times I \to X_1 \times \{1\} \cup A \times I$ (flip the hat upside down)

- Both def. retr's fix X_0, so have a h.e. rel X_0. [b/c h.e. transitive]

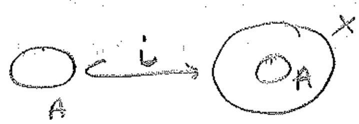
9/23 <u>Goal Thm</u>: (0.19) If (X,A) & (Y,A) both have HEP, & $f: X \to Y$ is a homotopy equivalence w/ $f|_A = \mathbb{1}$. Then f is a homotopy equivalence rel A.



(g fixes A, & $f \circ g$ & $g \circ f \xrightarrow{h} \mathbb{1}$ fix A)

<u>Cor</u>: If (X,A) has HEP & $i: A \hookrightarrow X$ is a h.equivalence, then A is a def. retr. of X.

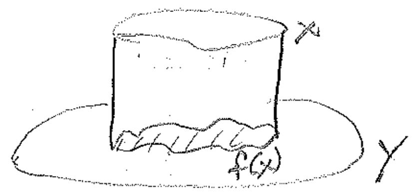
<u>Pf</u>:



(A,A) $\qquad$ (X,A) $\longrightarrow$ by thm, get h.e. that fixes A, so a def. retr.

<u>Cor</u>: $f: X \to Y$ is a h.e. $\Leftrightarrow$ X is a def. retr. of M_f (the mapping cylinder)

$$M_f = X \times I \sqcup Y /_{\sim}$$
$$(x,1) \sim f(x)$$



<u>Note</u>: • We already know Y is a def. retr. of M_f.
• Thus: $X \underset{he}{\simeq} Y \Leftrightarrow \exists Z$ that has both X & Y as def. retr's, & Z can be taken to be the mapping cylinder of any homotopy equivalence.

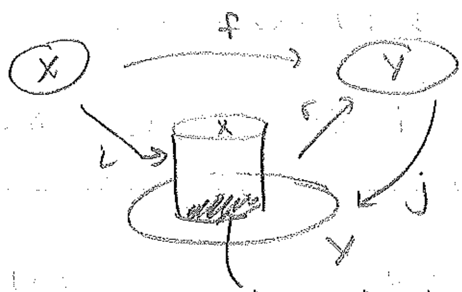
<u>Pf</u>:

i, j inclusions

$$\begin{tikzcd} & X \arrow[dl, "i"'] \arrow[dr, "f"] & \\ M_f \arrow[rr, "r"'] & & Y \arrow[ll, dashed, bend right, "j"'] \end{tikzcd}$$

<u>Note</u>: $f = r \circ i$

$i \underset{he}{\simeq} j \circ f$ (ie, diagram commutes up to homotopy)
(b/c $X \underset{he}{\simeq} Y$)



$j \circ f(X)$, but there's a def. retr. to X, the top.

Note: j & r are homotopy equivalences

<u>Claim</u>: i is a h.e. $\Leftrightarrow$ f is a h.e.

Pf: $i \underset{he}{\simeq} j \circ f$ & j a h.e. $\Rightarrow$ $i \Leftrightarrow f$ h.e's.:
Know that (M_f, X) has HEP (from mapping cyl. nbhds) b/c $X \times [0, 1/2]$ is a mapping cyl. nbhd.
So by 1^{st} Cor, if $i : X \to M_f$ a h.e, then X is a def. retr. of M_f. $\square$

Ch. 1: Fundamental Groups

<u>Def</u>: A <u>path</u> is a map $f : I \to X$, & if $f(0) = x_0$, $f(1) = x_1$, it's a path from x_0 to x_1.

<u>Def</u>: If f is a path from x_0 to x_1 & g is a path from x_1 to x_2, then the <u>concatination</u> of f & g is

$$f * g(s) = \begin{cases} f(2s), & s \in [0, 1/2] \\ g(2s-1), & s \in [1/2, 1] \end{cases}$$

, a path from x_0 to x_2.

<u>Def</u>: A <u>loop</u> is a path that starts & ends at the same pt. x_0, the <u>basepoint</u>.

<u>Note</u>: We have a set of loops starting at x_0 & an operation, $*$ (concatination of paths).