Math 722
Brian Street
www.math.wisc.edu
/~street/mat722

1/23 $\mathbb{R}$. Some polynomials don't have roots:
$x^2+1=0$

Formally define i. $i^2=-1$.
& $(-i)^2=-1$

$\alpha+i\beta$, $\alpha,\beta\in\mathbb{R}$.

A complex # is an ordered pair (α,β) of real #'s & we write $\alpha+i\beta$ instead of (α,β). As a set, the complex #'s, $\mathbb{C}$, is $\mathbb{R}^2$.

Addition: $(\alpha_1+i\beta_1)+(\alpha_2+i\beta_2):=(\alpha_1+\alpha_2)+i(\beta_1+\beta_2)$

$z_1+z_2=z_2+z_1$; $(z_1+z_2)+z_3=z_1+(z_2+z_3)$

Multiplication: $(\alpha_1+i\beta_1)(\alpha_2+i\beta_2):=(\alpha_1\alpha_2-\beta_1\beta_2)+i(\beta_1\alpha_2+\alpha_1\beta_2)$

$z_1z_2=z_2z_1$; $(z_1z_2)z_3=z_1(z_2z_3)$

$z_1(z_2+z_3)=z_1z_2+z_1z_3$

$\mathbb{C}$ is a <u>ring</u>.

Identify $\alpha\in\mathbb{R}$ w/ $\alpha+i0\in\mathbb{C}$, & $\mathbb{R}$ is a subring of $\mathbb{C}$.

$z=\alpha+i0$, we say z is "real".

If $z=0+i\beta$, we say z is "purely imaginary"

$0+i0=0$ is the only complex # which is both real & purely imaginary

$\alpha_1(\alpha_2+i\beta_2)=\alpha_1\alpha_2+i\alpha_1\beta_2$

$\Rightarrow$ $1=1+i0$ is the multiplicative identity of $\mathbb{C}$.

$\mathbb{C}$ is a <u>field</u>: Let $z=\alpha+\beta i\in\mathbb{C}$ with $z\neq 0$ (either $\alpha\neq 0 \vee \beta\neq 0$). Then $\exists$ a $z^{-1}\in\mathbb{C}$ w/ $z^{-1}z=zz^{-1}=1$

$z^{-1}=\frac{\alpha-i\beta}{\alpha^2+\beta^2}$ (check!)

$z = \alpha + i\beta \in \mathbb{C}$

α is called the real part of z ($\mathrm{Re}\, z$)

β is called the imaginary part of z ($\mathrm{Im}\, z$)

<u>Complex conjugate</u>: $\bar{z} = \alpha - i\beta$

$\bar{z} = z$ iff $\beta = 0$ iff z is real.

$\bar{\bar{z}} = z$

$\alpha = \mathrm{Re}\, z = \frac{\alpha + \alpha + i(\beta - \beta)}{2} = \frac{z + \bar{z}}{2}$

$\beta = \mathrm{Im}\, z = \frac{\alpha - \alpha + i(\beta + \beta)}{2i} = \frac{z - \bar{z}}{2i}$

$z\bar{z} = (\alpha + i\beta)(\alpha - i\beta) = \alpha^2 + \beta^2 + i(\alpha\beta - \alpha\beta) = \alpha^2 + \beta^2$

$|z|^2 = z\bar{z} = \alpha^2 + \beta^2$

$z^{-1} = \frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{\alpha - \beta i}{\alpha^2 + \beta^2}$

$|z| = \sqrt{z\bar{z}}$ = length of z as a vector

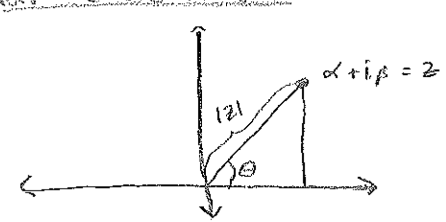
$|z_1 z_2|^2 = z_1 z_2 \bar{z}_1 \bar{z}_2 = z_1 \bar{z}_1 z_2 \bar{z}_2 = |z_1|^2 |z_2|^2 \Rightarrow |z_1 z_2| = |z_1| \cdot |z_2|$

$1 = |1| = |zz^{-1}| = |z||z^{-1}| \Rightarrow \frac{1}{|z|} = \left|\frac{1}{z}\right|$

Also: <u>Triangle Inequality</u>: $|z_1 + z_2| \leq |z_1| + |z_2|$

distance: $|z_1 - z_2| \Rightarrow \mathbb{C}$ is a metric sp so has the usual topology on $\mathbb{R}^2$

<u>Polar Coordinates</u>



$\theta \in [0, 2\pi) \leftarrow$ the <u>argument</u> of z

$\alpha = |z| \cos\theta$

$\beta = |z| \sin\theta$

$z_1 = r_1 \cos\theta_1 + i r_1 \sin\theta_1 = r_1(\cos\theta_1 + i \sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 z_2 = r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2))$

$\quad = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$

Old: $f: \mathbb{R} \to \mathbb{R}$ new: $f: \mathbb{C} \to \mathbb{C}$

$f(z)$

one variable, not 2

$f: \Omega \to \mathbb{C}$, ($\Omega \subseteq \mathbb{C}$ open)

or $f: \Omega \to \mathbb{R}$, ($\Omega \subseteq \mathbb{R}$ open), or a mix of the 2.

Def: The function $f(z)$ is said to have limit A as $z \to a$ ($\lim_{z\to a} f(z) = A$) if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < |z-a| < \delta$, $\Rightarrow |f(z) - A| < \varepsilon$.

$\lim_{z\to a} f(z) = A$, $z \in \mathbb{C}$, $A \in \mathbb{C}$, iff $\lim_{z\to a} \operatorname{Re} f(z) = \operatorname{Re} A$ &

$\lim_{z\to a} \operatorname{Im} f(z) = \operatorname{Im} A$

iff $\lim_{z\to a} \overline{f(z)} = \overline{A}$

$\lim_{z\to a} f_1(z) f_2(z) = \lim_{z\to a} f_1(z) \lim_{z\to a} f_2(z)$: products are cts!

1/25 HW#1: p. 28 # 1, 2, 4, 5, 7 due Fri.

$\lim_{z\to a} f(z) + g(z) = \lim_{z\to a} f(z) + \lim_{z\to a} g(z)$

$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{\lim_{z\to a} f(z)}{\lim_{z\to a} g(z)}$ if $\lim_{z\to a} g(z) \neq 0$.

Def: f is continuous at a if $\lim_{z\to a} f(z) = f(a)$, & f is continuous if f is cts at every pt of its domain.

→ If f & g are cont, then so are fg, $f+g$, f/g (if $g \neq 0$)

<u>Derivative</u> : $f:\Omega\to\mathbb{C}$, $\Omega\subseteq\mathbb{C}$ open, $a\in\Omega$

$$f'(a)=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\qquad (h\in\mathbb{C})$$

Product, Quotient & Chain rules as before:
$(fg)'=f'g+fg'$,
$(f\circ g)'=(f'\circ g)g'$, ...

<u>Ex</u>: Suppose $f:\mathbb{C}\to\mathbb{C}$, $\operatorname{Im} f\equiv 0$ (so f takes values in $\mathbb{R}$)
& suppose $f'(a)$ exists.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{\substack{h\to 0\\ h\in\mathbb{R}}}\frac{f(a+h)-f(a)}{h}\in\mathbb{R}$$

$$f'(a)=\lim_{\substack{h\to 0\\ h\in\mathbb{R}}}\frac{f(a+ih)-f(a)}{ih}=\lim_{\substack{h\to 0\\ h\in\mathbb{R}}}-i\cdot\frac{f(a+ih)-f(a)}{h}\in i\mathbb{R}$$

(ie, h is purely imag.)

$f'(a)=0$ (b/c only # in $\mathbb{R}$ & $i\mathbb{R}$ is 0)

- If $f:\mathbb{C}\to\mathbb{C}$ is diff. everywhere & $\operatorname{Im} f\equiv 0$, then $f\equiv c$.

- Differentiable $\Rightarrow$ continuous

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$

$$f(z+h)-f(z)=h\,\frac{f(z+h)-f(z)}{h}$$

$$\lim_{h\to 0} f(z+h)-f(z)=\left(\lim_{h\to 0} h\right)f'(z)=0\cdot f'(z)=0$$

$$\Rightarrow \lim_{h\to 0} f(z+h)=f(z)\qquad \text{(just like in calc)}$$

<u>Def</u>: Let $f:\Omega\to\mathbb{C}$. We say f is <u>analytic</u> (<u>holomorphic</u>) if the derivative of f exists at every pt of Ω.

- Analytic fcns are continuous.

$f(z) = u(z) + i v(z)$, $v, u : \Omega \to \mathbb{R}$ (v & u are cont & in fact diff) – not holo. fcns think of $u, v : \mathbb{R}^2 \to \mathbb{R}$

$z = x + iy$ (think of as (x,y))

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f((x+h)+iy) - f(x+iy)}{h} = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{u(x+h+iy) - u(x+iy)}{h} + i \frac{v(x+h+iy) - v(x+iy)}{h}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Also,

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(x + i(y+h)) - f(x+iy)}{ih}$$

$$= -i\left(\frac{\partial f}{\partial y}\right) = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- If $f = u + iv$ is analytic, the Cauchy-Riemann eqns hold. (the converse is true as well)

- If f is holomorphic, so is f'! As a consequence, u & v are C^∞ (from $\mathbb{R}^2 \to \mathbb{R}$). ($\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \Rightarrow$ order of partials doesn't matter)

- $\Delta = \partial/\partial x^2 + \partial/\partial y^2$, the Laplacian. $\Delta u = 0 = \Delta v$, ie u & v are harmonic. B/c:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \cdot \frac{\partial v}{\partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly for $\Delta v = 0$.

Def: If 2 harmonic fcns $u, v : \Omega \to \mathbb{R}$ ($\Omega \subseteq \mathbb{R}^2$) satisfy the Cauchy-R eqns, we say v is "the" harmonic conjugate of u. (v is unique up to constants) (u is the harmonic conjugate of $-v$).

Suppose $u, v : \Omega \to \mathbb{R}$ cont. w/ continuous 1st order partials which satisfy the C-R eqns.

Claim: $u + iv$ is holomorphic

Pf: $u(x+h, y+k) - u(x,y) = \frac{\partial u}{\partial x} h + \frac{\partial u}{\partial y} k + \varepsilon_1$ ($\varepsilon_1 \to 0$ faster than h,k)

$v(x+h, y+k) - v(x,y) = \frac{\partial v}{\partial x} h + \frac{\partial v}{\partial y} k + \varepsilon_2$

$f(z) = u(x,y) + iv(x,y)$

$$\Downarrow \quad \lim_{h+ik \to 0} \frac{\varepsilon_1}{h+ik} = 0$$

$$\lim_{h+ik \to 0} \frac{\varepsilon_2}{h+ik} = 0$$

$$f(z+h+ik) - f(z) = \frac{\partial u}{\partial x} h + \underbrace{\frac{\partial u}{\partial y} k}_{\text{C-R}} + i \frac{\partial v}{\partial x} h + i \frac{\partial v}{\partial y} k + \varepsilon_1 + \varepsilon_2$$

$$= \frac{\partial u}{\partial x}(h + ik) + i \frac{\partial v}{\partial x}(h + ik) + \varepsilon_1 + \varepsilon_2$$

$$\lim_{h+ik \to 0} \frac{f(z+h+ik) - f(z)}{h+ik} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad \square$$

Ex: $u = x^2 - y^2$ (harmonic). Find h. conj.

$\frac{\partial u}{\partial x} = 2y, \quad \frac{\partial u}{\partial y} = -2y \qquad v = ?$

Need $\frac{\partial v}{\partial x} = 2y \quad , \quad \frac{\partial v}{\partial y} = 2x$ by C-R eqns

$v = 2xy + \phi(y)$

$\frac{\partial v}{\partial y} = 2x + \phi'(y) \Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = c \in \mathbb{R}$

$\Rightarrow v = 2xy + c$

Given harmonic u, can find v s.t. $u + iv$ is holo.

1/28 $f:(\Omega \subseteq \mathbb{C} \text{ open}) \to \mathbb{C}$, $f(z)$, $z = x+iy$

$f(z) = f(x+iy) = u(x,y) + iv(x,y)$

$u,v : (\Omega \subseteq \mathbb{R}^2 \text{ open}) \to \mathbb{R}$

*f is holomorphic (analytic) iff $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. In this case, u & v are harmonic: $\Delta u = 0 = \Delta v$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, and f is infinitely diff, ie f' is holomorphic; & u,v are C^∞. Given $u: \Omega \to \mathbb{R}$ harmonic, $\exists !\, v: \Omega \to \mathbb{R}$ (up to constants) w/ u+iv holomorphic. v is called the harmonic conjugate of u.

$z = x+iy$, $\bar{z} = x - iy$.

$\Rightarrow \quad x = \frac{1}{2}(z+\bar{z})$, $y = -\frac{1}{2}i(z-\bar{z})$ $\qquad (1/i = -i)$

$f(x,y) = f(\frac{1}{2}(z+\bar{z}), -\frac{1}{2}i(z-\bar{z}))$ — use Chain Rule

$\frac{\partial f}{\partial z} := \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$ & $\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$

If f is holomorphic,

- $f'(z) = \frac{\partial f}{\partial z}(z)$
- $\frac{\partial f}{\partial \bar{z}} = 0$ ($\Leftrightarrow$ C-R eqns)

We think of holo. fcns as being fcns of one variable, z. They are independent of $\bar{z}$ b/c $\partial f/\partial \bar{z} = 0$.

of holo. fcns

Ex's: (1) Constant: $f(z) = c$: $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = 0$

(2) Identity: $f(z) = z$: $\lim_{h \to 0} \frac{z+h-z}{h} = 1$

(3) [NOT holo.] $f(z) = \bar{z} = x - iy$: $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ $(1 \neq -1)$

(fcn can only involve z's)

(4) Polynomials in z : $P(z) = a_n z^n + \cdots + a_0$, $a_i \in \mathbb{C}$

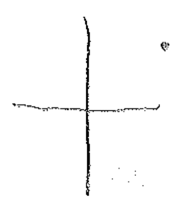
Polynomials: $P(z) = a_n z^n + \cdots + a_0$

- If $a_n \neq 0$, degree of P is n, $\deg P = n$.
- If $n > 0$, Fundamental Thm of Alg: $\exists\, \alpha_1 \in \mathbb{C}$ w/ $P(\alpha_1) = 0$.
  (we will prove this)
- If $P(\alpha_1) = 0$, $P(z) = (z - \alpha_1) P_1(z)$ where $\deg P_1 = n-1$
  $\Rightarrow$ $P(z) = a_n (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$. α_i not nec. distinct.

- If h of the α_j coincide, we say P has a root (zero) of order h at α_j.
- # of the zeros counted w/ multiplicity is the degree of P.

$\mathbb{R}$

$-\infty \quad \longleftrightarrow \quad +\infty$ (add 2 pts to compactify $\mathbb{R}$)

$\mathbb{C}$  $\cdot\,\infty$ add one pt; b/c can go to ∞ infinitely many ways $\Rightarrow$ wraps up into sphere & ∞ = north pole

- $P(\infty) = \infty$ if $\deg P > 0$.

Rational Function: $R(z) = P(z)/Q(z)$ are holo. (where $Q \neq 0$)

- We assume P & Q have no common factors (ie no common zeros)
- When $Q(z) = 0$, define $R(z) = \infty$. Poles are where $Q(z) = 0$
- $R(z)$ is holo. on the open set $\{Q \neq 0\}$.
- $R'(z) = \dfrac{P'(z)Q(z) - Q'(z)P(z)}{Q(z)^2}$, still rational. Has poles

of one greater order (α of order h: $Q(z)^2$ order $2h$, $Q'(z)$ order $h-1$, $Q(z)$ order h $\Rightarrow$ $h-1/2h$ $\Rightarrow$ $h+1$)

Power Series : $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

- In real, $f(x) = e^{-1/x^2}$, but Taylor series $= 0$, so doesn't converge to f.
- In complex, always converges, $\tilde{\varepsilon}$, to f.

Topics from Real Analysis

① Sequences : $\{a_n\}_{n=1}^{\infty}$ complex #'s.

- $\lim_{n\to\infty} a_n = A$ if $\forall \varepsilon > 0\ \exists N$ s.t. $n \geq N \Rightarrow |a_n - A| < \varepsilon$.
- $\lim_{n\to\infty} a_n = \infty$ if $\forall M\ \exists N$ s.t. $n \geq N \Rightarrow |a_n| > M$

- A sequence is Cauchy if $\forall \varepsilon > 0\ \exists N$ s.t. $n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon$
- $\mathbb{C}$ is complete, ie a sequence is convergent iff it is Cauchy.

② Series : $a_1 + a_2 + \cdots = \sum_{j=1}^{\infty} a_j$, $a_j \in \mathbb{C}$.

Let $s_n = \sum_{j=1}^{n} a_j$. We say $\sum_{j=1}^{\infty} a_j$ converges if $\lim_{n\to\infty} s_n$ exists. In this case $\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} s_n$.

(if $\lim s_n = \infty$, we say series diverges to ∞)

- If $\sum_{j=1}^{\infty} |a_j|$ converges, so does $\sum_{j=1}^{\infty} a_j$.

Pf : $\left| \sum_{j=1}^{n} |a_j| - \sum_{j=1}^{m} |a_j| \right| = \sum_{j=n\wedge m}^{n\vee m} |a_j| \geq \left| \sum_{j=n\wedge m}^{n\vee m} a_j \right|$ (max, min)

$= \left| \sum_{j=1}^{n} a_j - \sum_{j=1}^{m} a_j \right|$, so if $\{|a_j|\}$ Cauchy, a_j Cauchy.

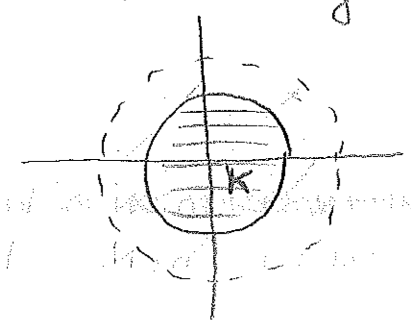
So abs. conv. $\Rightarrow$ conv. (works in all Banach sp's, only used completeness & Δ-ineq).

- Uniform Convergence

$f_n : E \to \mathbb{C}$. We say $f_n(z) \to f(z)$ uniformly (in z) if $\forall \varepsilon > 0\ \exists N$ s.t. $\forall z \in E$, $n \geq N \Rightarrow |f_n(z) - f(z)| < \varepsilon$.

(unif. conv. seq. are Cauchy & conversely)

- Uniform limits of cts fcns are cts.
  $f_n : (\Omega \subseteq \mathbb{C}) \to \mathbb{C}$, open. We say $f_n \to f$ uniformly on cpt subsets if $\forall K \subseteq \Omega$ cpt, $f_n(z) \to f(z)$ uniformly on K.



→ close to boundary, may have to choose larger N, ie, converges slower near boundary.

→ If $f_n \to f$ unif. on cpt sets & f_n are holo, then f is holo.

1/30 $\sum_{j=1}^{\infty} a_j$ → sums of complex #s. $f_j(z) : \mathbb{C} \to \mathbb{C}$, $\sum_{j=1}^{\infty} f_j(z)$ → sums of fcns.

Suppose for $n \geq N$, $|f_n(z)| \leq M\overset{\mathbb{R}}{a_n}$ & $\sum_{n=N}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly.

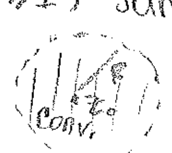
$M_2 \geq M_1$, $\sum_{n=1}^{M_1} |f_n(z)| - \sum_{n=1}^{M_2} |f_n(z)| = \sum_{n=M_1}^{M_2} |f_n(z)| \leq M \sum_{n=M_1}^{M_2} a_n \xrightarrow{M_1, M_2 \to \infty} 0$, b/c a_n Cauchy, & $\sum_{M_1}^{M_2}$ is diff. of partial sums. (Weierstrauss M-test)

Power Series

- $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, $a_n \in \mathbb{C}$, $z_0 \in \mathbb{C}$
- $f(z + z_0) = \sum_{n=0}^{\infty} a_n z^n$

Consider $1 + z + z^2 + \cdots$
- Partial Sum: $1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z}$ (mult. by $1-z$)
  $\xrightarrow{n \to \infty} \frac{1}{1-z}$ if $|z| < 1$

  If $|z| \geq 1$, sum diverges.



div.

Thm: $\sum_{n=0}^{\infty} a_n z^n$. There is a number R, $0 \le R \le \infty$, called the radius of convergence, w/ the following props:

(1) The series conv. abs. for $|z|<R$, & if $0 \le \rho < R$, the series conv. unif. for $|z| \le \rho$. (uniformly on cpt sets)

(2) For $|z|>R$, the terms of the series are unbdd & the series diverges.

(3) On $|z|<R$, the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic. It's derivative is $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$.

- $1/R = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}$

Pf: (1) If $|z|<R$, let ρ, ρ' be s.t. $|z| \le \rho < \rho' < R$.

$1/\rho' > 1/R \Rightarrow \exists\, n_0$ s.t. $n \ge n_0$, $|a_n|^{1/n} < 1/\rho'$, ie,

$|a_n| < \frac{1}{(\rho')^n}$. For $n > n_0$,

$|a_n z^n| < \frac{|z|^n}{(\rho')^n} \le \left(\frac{\rho}{\rho'}\right)^n$

a term of a geom. series b/c <1.

$\Rightarrow$ conv. abs. & uniformly for $|z| \le \rho$.

(2) If $|z|>R$, pick ρ w/ $|z| > \rho > R$. $1/\rho < 1/R$. $\exists$ arbitrarily large n w/ $1/\rho < \sqrt[n]{|a_n|} \Rightarrow 1/\rho^n < |a_n|$. ($\infty$-ly many n, s.t. $\sqrt[n]{|a_n|} > 1/\rho$)

$\Rightarrow |a_n z^n| > \frac{1}{\rho^n} \cdot |z|^n = \left(\frac{|z|}{\rho}\right)^n \to \infty$ as $n \to \infty$.

$\Rightarrow$ div.

(3) $\frac{d}{dz}\left(\sum_{n=0}^{\infty} a_n z^n\right) \overset{?}{=} \sum_{n=1}^{\infty} n a_n z^{n-1}$

Fact: $\sqrt[n]{n} \to 1$, so these 2 series have the same radius of convergence.

$f(z) = \sum_{n=0}^{\infty} a_n z^n$. For $|z|<R$, $f(z) = \underbrace{\sum_{k=0}^{n-1} a_k z^k}_{\text{sum}} + \underbrace{\sum_{k=n}^{\infty} a_k z^k}_{\text{remainder}} =: S_n(z) + R_n(z)$

$f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. $\lim_{n\to\infty} S_n'(z) = f_1(z)$ b/c S_n a poly.

want $f'(z) = f_1(z)$.

$\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0)$, for z near z_0, want this to be small.

$= \frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0) + (S_n'(z_0)-f_1(z_0)) + \frac{R_n(z)-R_n(z_0)}{z-z_0}$

1. Fix $|z|,|z_0|<\rho<R$, $\varepsilon>0$.

$\frac{R_n(z)-R_n(z_0)}{z-z_0} = \sum_{k=n}^{\infty} a_k \frac{z^k-z_0^k}{z-z_0} = \sum_{k=n}^{\infty} a_k(z^{k-1}+z^{k-2}z_0+\dots+zz_0^{k-2}+z_0^{k-1})$

$|a_k(z^{k-1}+z^{k-2}z_0+\dots+z_0^{k-1})| \le |a_k| k\cdot\rho^{k-1}$

↑ term from conv. series

$\sum_{k=0}^{\infty} |a_k| k \rho^{k-1}$ conv, so tail is small.

$a_k^{1/k}$ 1 ρ

(ie $\sum_{n}^{\infty}$)

So if n suff. large, $\left|\sum_{k=n}^{\infty} a_k \frac{z^k-z_0^k}{z-z_0}\right| < \varepsilon$.

$\lim_{n\to\infty} S_n'(z_0) = f_1(z_0)$. so if n suff. large, $|S_n'(z_0)-f_1(z_0)|<\varepsilon$.

$\lim_{z\to z_0} \frac{S_n(z)-S_n(z_0)}{z-z_0} = S_n'(z_0)$, so if n is fixed as above,

$\exists\,\delta$ s.t. $|z-z_0|<\delta$, $\left|\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)\right|<\varepsilon$

$\Rightarrow \forall \varepsilon>0\ \exists\delta>0$ s.t. $0<|z-z_0|<\delta \Rightarrow \left|\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0)\right| < 3\varepsilon$.

So f diff, & derivative is f_1.

- $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

$f(0) = a_0$

$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \Rightarrow f'(0) = a_1$

$f''(z) = 2a_2 + 6a_3 z + 12 a_4 z^2 + \cdots \Rightarrow f''(0) = 2a_2$

If f is given by a power series, then

$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2}\cdot z^2 + \dots + \frac{f^{(n)}(0)}{n!} z^n + \cdots$

- Every holo. fcn is given "locally" by power fcn.

- Conv for $|z|<R$
- Div. for $|z|>R$
- When $|z|=R$, lots of stuff...

<u>Special Fcns</u>

- $f(z)=e^z$; $f:\mathbb{C}\to\mathbb{C}$

<u>Def</u>: $f(z)=e^z$ is the fcn. s.t. $f'(z)=f(z)$ & $f(0)=1$.

$f(z)=a_0+a_1z+\cdots+a_nz^n+\cdots$
$f'(z)=a_1+2a_2z+\cdots+na_nz^{n-1}+\cdots$
$\Rightarrow a_{n-1}=na_n,\quad a_0=1 \quad\Rightarrow a_n=\frac{1}{n!}$
So, $e^z=\sum_{n=0}^{\infty} z^n/n!$.

$\sqrt[n]{n!}\to\infty = R$ as $n\to\infty$ $\Rightarrow$ sum conv. $\forall z$, ie, conv. unif. on all cpt. sets.

<u>Ex</u>: $e^{z_0+z}\cdot\frac{1}{e^{z_0}}=g(z)$. $\quad g(0)=1$
$g'(z)=\frac{1}{e^{z_0}}\cdot e^{z_0+z}=g(z)$
$\Rightarrow g(z)=e^z \Rightarrow e^{z_0+z}=e^{z_0}e^z$ (diff. eqn def. makes these props. easy to prove)

*So, e^z has an inverse: $e^ze^{-z}=e^0=1$, e^z is never 0.
(but it takes all other values, unlike w/ $\mathbb{R}$ #'s).

* $\overline{e^z}=\overline{\sum_{n=0}^{\infty} z^n/n!}=\sum_{n=0}^{\infty}\frac{\overline{z^n}}{n!}=\sum_{n=0}^{\infty}\frac{\bar{z}^n}{n!}=e^{\bar{z}}$
(true for Taylor series w/ all coeffs real: $\overline{f(z)}=f(\bar{z})$)

* Let $y\in\mathbb{R}$.
$|e^{iy}|^2=e^{iy}\cdot\overline{e^{iy}}=e^{iy}e^{-iy}=1$ (for pure im, on the circle)
If $x\in\mathbb{R}$, $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

- $\cos(z)=\frac{e^{iz}+e^{-iz}}{2}$, $\quad\sin(z)=\frac{e^{iz}-e^{-iz}}{2i}$

2/1 HW#2: p.41/1,3,8; p.44/1,2; p.47/6,7

Recall: e^z, $z \in \mathbb{C}$. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ (defined (converges) $\forall z$)

- $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

- If $z \in \mathbb{R}$, these all agree w/ classical definitions

- $\cos^2 z + \sin^2 z = 1$, but $\cos z$ & $\sin z$ are large if $\operatorname{Im}(z)$ is large (unlike in $\mathbb{R}$)

- $e^{iz} = \cos z + i\sin z$

Let $y \in \mathbb{R}$. Then $e^{iy} = \cos y + i\sin y$, both real fcns

Let $x, y \in \mathbb{R}$. Then $e^{x+iy} = e^x(\cos y + i\sin y)$ (all real fcns)

$|e^{iy}| = \sqrt{\cos^2 y + \sin^2 y} = 1$ (any # w/ modulus 1 can be written e^{iy})

$e^{x+iy} = e^x \cdot e^{iy}$

$\Downarrow$ $\in (0,\infty)$ $\uparrow \in \mathbb{C}$ of modulus 1

$|e^{x+iy}| = e^x$, so $0 \neq e^z$ for any z.

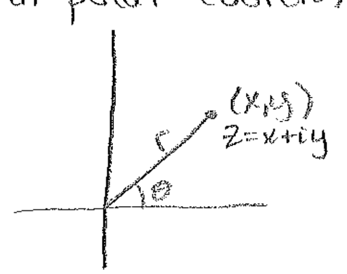
- $e^{iy} = e^{i(y+2\pi n)}$, $n \in \mathbb{Z}$, so $z \mapsto e^z$ is not injective, & it's not surjective (b/c of 0), but that's the only failure. Every nonzero $\mathbb{C}$ number is e^z for some z.

$z \neq 0$, $z = e^{\omega + i\theta}$, $\omega, \theta \in \mathbb{R}$.

$z = e^{\omega}(\cos\theta + i\sin\theta)$

$= e^{\omega}\cos\theta + ie^{\omega}\sin\theta$

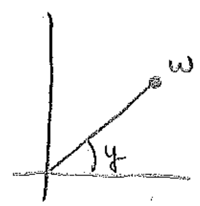
- If, in polar coords, then $r = e^{\omega}$, & $e^{\omega+i\theta} = z$



Logarithms

$z = \log w$ means $e^z = w$. Every w has a Logarithm ($w \neq 0$). In fact, we can change z by a mult of $2\pi i$ & obtain another log.

$z = x + iy$, & $e^{x+iy} = w$
$e^x e^{iy} = w \Rightarrow e^x = |w| \Rightarrow x = \log|w|$ ← real log
(b/c $|e^{iy}| = 1$)
$e^{iy} = w/|w| \in \mathbb{C}$ of modulus 1.



We call y the <u>argument</u> of w, $\arg w$, defined only up to mult. of 2π.

So, $\operatorname{Log} w = \log|w| + i \arg(w)$
↑ you have to make a choice.

$a, b \in \mathbb{C}$, we can now define a^b.

$a^b := e^{b \log a}$ $(a \neq 0)$, but this depends on our def of log (arg), b/c im. part of log can vary by $2\pi n$, $n \in \mathbb{Z}$.

- If $b \in \mathbb{Z}$, then $e^{b \log a} = a^b$ is unambiguously defined.

Conformal

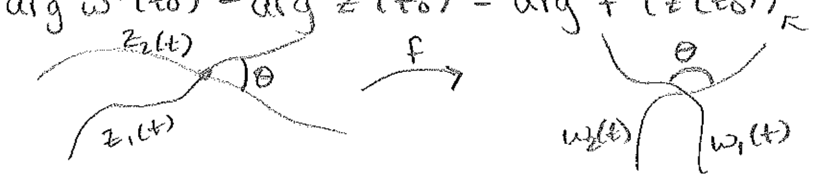
$z : [a,b] \to \Omega$ ($\subseteq \mathbb{C}$ open.), $z'(t)$, $f : \Omega \to \mathbb{C}$ analytic.

$w(t) = f \circ z(t)$.

$w'(t) = f'(z(t)) \cdot z'(t)$. Fix t_0, w/ $z'(t_0) \neq 0$ & $f'(z(t_0)) \neq 0$, so $w'(t_0) \neq 0$.

$\arg w'(t_0) = \arg(f'(z(t_0))\, z'(t_0))$
$= \arg(f'(z(t_0))) + \arg(z'(t_0))$ [mod 2π]

$\Rightarrow \arg w'(t_0) - \arg z'(t_0) = \arg f'(z(t_0))$ ← doesn't depend on z, just f
$\Rightarrow$ angle remains unchanged

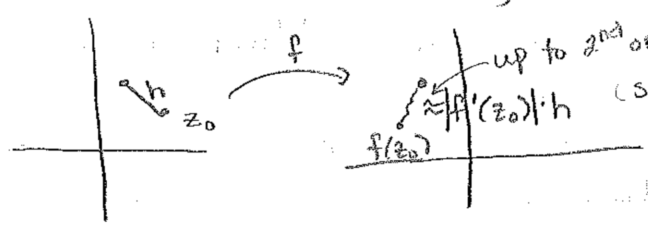


"f preserves angles."

<u>Def</u>: f is <u>conformal</u> if $\arg(w'(t_0)) - \arg(z'(t_0))$
$= \arg(f'(z(t_0)))$, ie, if f preserves angles.

- $\lim_{z \to z_0} \dfrac{|f(z) - f(z_0)|}{|z - z_0|} = |f'(z_0)|$

In the limit, a small line seg starting at h, is



up to 2^{nd} order term (h^2)

scaled by factor of $|f'(z_0)|$, but doesn't matter what direction the line seg went.

- If ↗ this is true then either f is analytic or $\overline{f(z)}$ is analytic.
- Also, conformal ⇔ analytic.

<u>Pf</u>$^{(\Rightarrow)}$: $f(z) = x + iy$, & assume $\partial f/\partial x$ & $\partial f/\partial y$.

$w(t) = f(z(t))$, $z(t) = x(t) + iy(t)$

$\frac{d}{dt} w(t) = \frac{\partial f}{\partial x}(z(t))\, x'(t) + \frac{\partial f}{\partial y}(z(t))\, y'(t)$. If angles are preserved, ie f is conformal, then

$\arg\left(\frac{w'(t_0)}{z'(t_0)}\right)$ must be independent of $\arg(z'(t_0))$.

$$\frac{d}{dt} w(t) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) z'(t) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\overline{z'(t)}$$

Divide by $z'(t_0)$:

$$\frac{w'(t_0)}{z'(t_0)} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\frac{\overline{z'(t_0)}}{z'(t_0)}$$

(← has const. arg, so coeff. of $\frac{\overline{z'(t_0)}}{z'(t_0)}$ must be zero.)

we conclude that $\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0 = \frac{\partial f}{\partial \bar{z}} \Rightarrow$ CR eqns

⇒ holo.

Linear Fractional Transformations

$S(z) = \frac{az+b}{cz+d}$, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$

- functions on $\mathbb{C} \cup \{\infty\}$ (continuous map on Riemann sphere)
- $S(\infty) := a/c$ & $\frac{\text{non-zero}}{\text{zero}} := \infty$.

For $r \neq 0$

$\frac{az+b}{cz+d} = \frac{(ra)z+rb}{(rc)z+rd}$, so change a,b,c,d by r, & get same fcn.

WLOG, we always assume $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$.

Let $z = z_1/z_2$, $\frac{w_1}{w_2} = w = S(z)$.

$S(z) = \frac{a\, z_1/z_2 + b}{c\, z_1/z_2 + d} = \frac{az_1 + bz_2}{cz_1 + dz_2}$

$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Rightarrow \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

So composition of LFT's can be done trivially:

$T_1(z) = \frac{a_1z+b_1}{c_1z+d_2}$, $T_2(z) = \frac{a_2z+b_2}{c_2z+d_2}$, then

$T_1 \circ T_2$ is a LFT & corresp's to $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

$T(z) = \frac{az+b}{cz+d} \Rightarrow T^{-1}(z) = \frac{dz-b}{-cz+a}$ (remember, $\det T = 1$)

b/c $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ if $\det = 1$.

$SL_2(\mathbb{C})$ = gp of matrices w/ det 1, but I & $-I$ give id map

$SL_2(\mathbb{C})/\{\pm I\} \cong$ Linear fractional transformations.

2/4 Last time: $S(z) = \frac{az+b}{cz+d}$; $S \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$ (or $= 1$)

$S: \mathbb{C} \cup \{\infty\} \circlearrowleft$

$\Downarrow$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$

3 degrees of freedom: Given 3 distinct pts, $z_2, z_3, z_4 \in \mathbb{C} \cup \{\infty\}$,

Claim: $\exists!$ LFT which carries them to $1, 0, \infty$, resp.

Pf: (Existence): If none of z_2, z_3, z_4 are ∞, then

$$S(z) = \frac{z - z_3}{z - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}.$$

If $z_2 = \infty$, $S(z) = \frac{z - z_3}{z - z_4}$; If $z_3 = \infty$ $\frac{z_2 - z_4}{z - z_4}$; If $z_4 = \infty$ $\frac{z - z_3}{z_2 - z_3}$

(Uniqueness): If T were another LFT that takes $z_2, z_3, \text{\&}\ z_4$ to $1, 0, \infty$, then $S \circ T^{-1}$ fixes $1, 0, \infty$, but $S \circ T^{-1} = \frac{az+b}{cz+d}$. Plug in "fixes $1, 0, \infty$," solve, & see $S \circ T^{-1}(z) = z$.

Def: The cross ratio $(z_1, z_2, z_3, z_4) \in \mathbb{C} \cup \{\infty\}$ is the image of z_1 under the linear transf. which carries (z_2, z_3, z_4) to $(1, 0, \infty)$

Thm: If z_1, z_2, z_3, z_4 are distinct & T is a LFT, then (Tz_1, Tz_2, Tz_3, Tz_4) is the same as (z_1, z_2, z_3, z_4).

Pf: If $S(z) = (z, z_2, z_3, z_4)$, then $S \circ T^{-1}$ takes Tz_2, Tz_3, Tz_4 to $1, 0, \infty$. Thus, (Tz_1, Tz_2, Tz_3, Tz_4) $= S \circ T^{-1}(Tz_1) = S(z_1) = (z_1, z_2, z_3, z_4)$

S takes z_2, z_3, z_4 to $1, 0, \infty$, T takes w_2, w_3, w_4 to $1, 0, \infty$, then $T^{-1} \circ S$ takes z_2, z_3, z_4 to w_2, w_3, w_4, so can use any 3 pairs of pts, not just $1, 0, \infty$.

Thm: The cross ratio (z_1, z_2, z_3, z_4) is real iff the four pts lie on a circle or a straight line.

Pf: WTS the image of the real line under a LFT is either a circle or a straight line.

The value of $w = T^{-1}(z)$ for $z \in \mathbb{R}$: $T(w) = \overline{T(w)}$.

So, $\frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}$

$(a\bar{c} - c\bar{a})\underset{w\bar{w}}{|w|^2} + (a\bar{d} - c\bar{b})w + (b\bar{c} - d\bar{a})\bar{w} + b\bar{d} - d\bar{b} = 0$

If $a\bar{c} - c\bar{a} = 0$, then $a\bar{d} - c\bar{b} \neq 0$, b/c $a\bar{d} - c\bar{b} = \bar{a}d - \bar{c}b$. If $a \neq 0$, mult. by a to get $\bar{a}(ad - bc) \neq 0$. If $c \neq 0$, mult. by c to get $\bar{c}(ad - bc) \neq 0$. So, this is a line. (either $a \neq 0$ or $c \neq 0$, since $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$).

If $a\bar{c} - c\bar{a} \neq 0$, divide by it. Complete the square, then $\left| w + \frac{\bar{a}d - \bar{c}b}{\bar{a}c - \bar{c}a} \right| = \left| \frac{ad - bc}{\bar{a}c - \bar{c}a} \right|$ is a circle

(↑center) (↑radius)

Think of a line as a circle through ∞. So LFT's take "circles to circles".

$SL(2, \mathbb{R})$ preserves the real line & the unit circle.

Integration Over Complex Numbers

Line Integrals

Let $f: [a,b] \to \mathbb{C}$, $f(t) = u(t) + i v(t)$.

- $\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt$

- $\int_a^b c f(t)\,dt = c \int_a^b f(t)\,dt$
- $\int_a^b f(t) + g(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$

- $\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt$

Path Integrals: $f:\Omega \subseteq \mathbb{C} \to \mathbb{C}$

$z(t):[a,b]\to\mathbb{C}$, where z parameterizes γ. z is diff. at all but finitely many pts.

Def: $\int_\gamma f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$

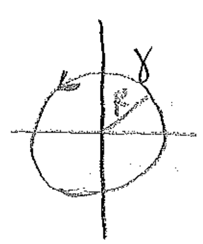
If $t=t(\tau)$ increasing maps $\alpha \le \tau \le \beta$ to $a \le t \le b$, then

$\int_a^b f(z(t))\,z'(t)\,dt = \int_a^b f(z(t(\tau)))\,z'(t(\tau))\,t'(\tau)\,d\tau$ &

$\frac{d}{d\tau} z(t(\tau)) = z'(t(\tau))\,t'(\tau)$

- def. of path integral does not depend on the speed with which you move along the path.
- The opposite arc, $-\gamma$, has parameterization $z(-t)$, $-b \le t \le -a$, and

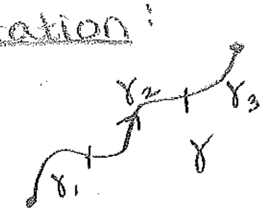
$$\int_{-\gamma} f(z)\,dz = \int_{-b}^{-a} f(z(-t))(-z'(-t))\,dt = \int_b^a f(z(t))\,z'(t)\,dt$$
$$= -\int_\gamma f(z)\,dz.$$



$\int_\gamma f(z)\,dz$. You can parameterize it any way you like.

Notation:

$\gamma = \gamma_1 + \gamma_2 + \gamma_3$



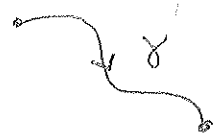
- $\int_\gamma f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz + \int_{\gamma_3} f\,dz$

$d\bar{z}$: $\int_\gamma f\,d\bar{z} := \overline{\int_\gamma \bar{f}\,dz}$

$\int_\gamma f\,dx = \int_a^b f(z(t))\,x'(t)\,dt = \frac{1}{2}\left(\int_\gamma f\,dz + \int_\gamma f\,d\bar{z}\right)$

$(z(t) = (x(t), y(t)))$ & $\int_\gamma f\,dy = \frac{1}{2i}\left(\int_\gamma f\,dz - \int_\gamma f\,d\bar{z}\right)$

$dz = dx + i\,dy$ & $d\bar{z} = dx - i\,dy$.

2/6 Last time:  γ $z(t),\ a \le t \le b$

(f cts)

$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$

$z(t) = (x(t), y(t))$

- $\int_{-\gamma} f = -\int_\gamma f$
- $\int_\gamma f(x,y)\,dx = \int_0^b f(x(t),y(t))\,x'(t)\,dt$ (similarly for dy)
- $\int_\gamma f\,d\bar{z} = \overline{\int_\gamma \bar{f}\,dz}$
- $\int_\gamma f\,dx = \frac{1}{2}\left(\int_\gamma f\,dz + \int_\gamma f\,d\bar{z}\right)$
  $\int_\gamma f\,dy = \frac{1}{2i}\left(\int_\gamma f\,dz - \int_\gamma f\,d\bar{z}\right)$

$\Rightarrow dz = dx + i\,dy$, $d\bar{z} = dx - i\,dy$.

If $f = u + iv$, $\int_\gamma f\,dz = \int_\gamma (u+iv)(dx + i\,dy) = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx)$

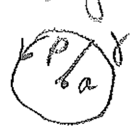
$\left|\int_\gamma f(z)\,dz\right| \le ?$ (estimate the integral)

Arc Length: $\int_\gamma f\,ds = \int_\gamma f\,|dz|$, with param. $z(t)$ $a \le t \le b$

$:= \int_a^b f(z(t))\,|z'(t)|\,dt$

- this is independent of the parameterization
- $\int_{-\gamma} f\,|dz| = \int_\gamma f\,|dz|$

- $\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right| \le \int_a^b |f(z(t))|\,|z'(t)|\,dt$
  $= \int_\gamma |f(z)|\,|dz|$, so integral over arc length can bound a path integral.
- $\int_\gamma |dz| =$ length of γ.

Ex: 

$\int_\gamma |dz| = \int_0^{2\pi} |z'(t)|\,dt = \int_0^{2\pi} |\rho i e^{it}|\,dt$

$= \rho\int_0^{2\pi} dt = 2\pi\rho$

$z(t) = a + \rho e^{it}$

$0 \le t \le 2\pi$

want to be small

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \int_\gamma \left(\max_{z\in\gamma}|f(z)|\right)|dz|$$

$$\le \left(\max_{z\in\gamma}|f(z)|\right)\cdot(\text{length of } \gamma)$$

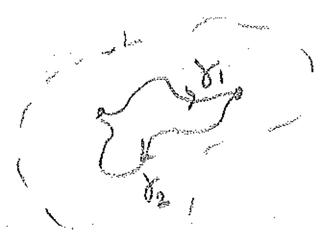
$\int_\gamma p\,dx + q\,dy$, p & q are fixed fcns on Ω

so given γ, get a number.

$\Omega \subseteq \mathbb{R}^2$ open & ctd.

Q: For what p & q does $\int_\gamma p\,dx + q\,dy$ depend only on the endpts of γ?

Comment:



$$\int_{\gamma_1 - \gamma_2} = \int_{\gamma_1} + \int_{-\gamma_2} = \int_{\gamma_1} - \int_{\gamma_2}$$

So, $\int_\gamma$ depends only on the endpts $\Leftrightarrow$ $\int_\gamma = 0$ for every closed loop γ.

Thm: The line integral $\int_\gamma p\,dx + q\,dy$ depends only on the endpts of γ iff $\exists$ fcn $u(x,y)$ w/ $p = \partial u/\partial x$ & $q = \partial u/\partial y$.

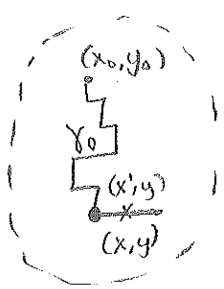
Pf: Sufficiency: Suppose $p = \partial u/\partial x$ & $q = \partial u/\partial y$. Then

$$\int_\gamma p\,dx + q\,dy = \int_a^b \frac{\partial u}{\partial x}(x(t),y(t))\,x'(t) + \frac{\partial u}{\partial y}(x(t),y(t))\,y'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\,u(x(t),y(t))\,dt = u(x(b),y(b)) - u(x(a),y(a))$$

($\Rightarrow$): Suppose $\int_\gamma p\,dx + q\,dy$ depends only on endpts of γ. Fix $(x_0,y_0) \in \Omega$. For any $(x,y) \in \Omega$, connect (x_0,y_0) to (x,y) by some path $\gamma_{(x,y)}$. Define $u(x,y) = \int_\gamma p\,dx + q\,dy$. (our assumption is that this is well-defined.)

Goal: Show $\partial u/\partial x = p$. Fix (x,y) & connect to (x_0,y_0) by a polygonal path that ends in a horizontal line:



$$u(x,y) = \underbrace{\text{const.}}_{\int_{\gamma_0} p\,dx + q\,dy} + \underbrace{\int_{\text{lower bd}}^{x} p\,dx}_{\text{horiz path} \Rightarrow \text{int. only in } x}$$

$\Rightarrow \frac{\partial u}{\partial x} = p(x,y)$. reverse roles of x & y,
& $\partial u/\partial y = q(x,y)$. □

Notation: • $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$.

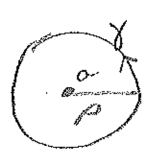
- If $p\,dx + q\,dy = du$ for some u, we say $p\,dx+q\,dy$ is exact. ($\Leftrightarrow$ every path int. depends only on the endpts)
- du determines u up to a constant.

When is $f(z)dz$ exact?
$f(z)dz = f(z)(dx + i\,dy) = f(z)dx + if(z)dy$
This is exact iff $\exists\, F(x,y)$ with $\frac{\partial F}{\partial x}(z) = f(z)$ &
$\partial F/\partial y(z) = if(z)$. $\Rightarrow \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$. This is the C-R eqns for F. $\Leftrightarrow$ F is analytic. In this case,
$f(z) = F'(z)$.

$\int_\gamma f(z)dz$ depends only on the endpts of γ iff $f = F'$ for some analytic F.

Ex: (a) If γ is a closed curve,
$\int_\gamma (z-a)^n dz = 0$ b/c $(z-a)^n = \frac{d}{dz}\frac{(z-a)^{n+1}}{n+1}$

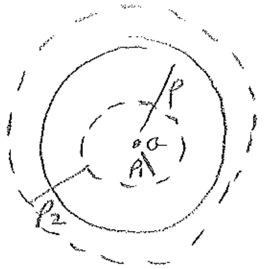
(b) If $n \in \mathbb{Z}$, $n \neq -1$, $(z-a)^n$, γ a closed loop not passing through a, then $\int_\gamma (z-a)^n dz = 0$.

(c) 

$z(t) = a + \rho e^{it}$, $0 \le t \le 2\pi$
$z'(t) = \rho i e^{it}$

$$\int_\gamma \frac{1}{z-a}dz = \int_0^{2\pi} \frac{1}{\rho e^{it}} \cdot \rho i e^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

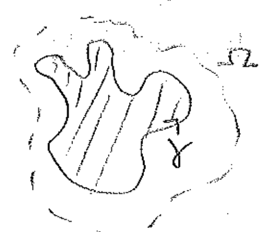
* doesn't depend on ρ & $\neq 0$.



$\rho_1 < |z-a| < \rho_2$. There is no analytic choice of $\log(z-a)$.

$\int_\gamma \frac{1}{z-a}\,dz = 0$

Can define $\log(z-a)$ for the plane minus a half-line centered at 0.

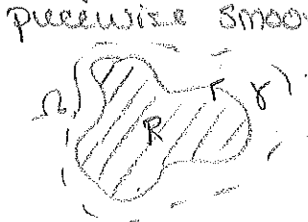


Cauchy's Thm (informal) f analytic on Ω, which contains interior of γ. Then $\int_\gamma f\,dz = 0$.

* problem w/ $\frac{1}{z-a}$, is that it's not analytic at a, so not on entire interior of γ.

2/8 p. 78/2,4 ; p. 80/1 ; p. 108/2,3,4

Cauchy's Thm (informal) : Region R. R = inside of a piecewise smooth closed curve γ.



f analytic on R means $\exists$ nbhd Ω of R w/ f analytic on Ω.

$\int_{\partial R = \gamma} f(z)\,dz = 0$.

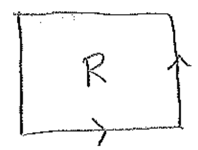
$f(z) = u(x,y) + iv(x,y)$ $(z = x+iy)$. We will see f analytic $\Rightarrow u \,\&\, v \in C^\infty$.

If $u, v \in C^1$, Cauchy's Thm easy (Green's Thm).

$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists $\forall z_0$: this is all we're assuming, so we don't know u, v are even C^1.

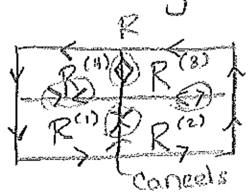
(Goursat proved this for u, v not assumed to be nice)

Start w/ R=rectangle



Thm: If f is analytic on R, then $\int_{\partial R} f(z)\,dz = 0$.

Pf: Let $\eta(R) = \int_{\partial R} f(z)\,dz$. Divide R into 4 $\cong$ sub-rectangles



$$\eta(R) = \int_{\partial R} f(z)\,dz = \int_{\partial R^{(1)}} f(z)\,dz + \int_{\partial R^{(2)}} f(z)\,dz + \int_{\partial R^{(3)}} f(z)\,dz + \int_{\partial R^{(4)}} f(z)\,dz$$

$$= \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)})$$

There is a j with $|\eta(R^{(j)})| \geq \frac{1}{4}|\eta(R)|$. Pick one & call it $R^{(1)}$.

Repeating, we get $R \supseteq R_1 \supseteq R_2 \supseteq R_3 \supseteq \cdots$ w/ each 1/4 of the previous.

$$|\eta(R_n)| \geq \tfrac{1}{4}|\eta(R_{n-1})|$$
$$\geq 4^{-n}|\eta(R)|$$

The rectangles R_n converge to a pt z^* in the sense that $\forall \delta > 0\ \exists N;\ n \geq N \Rightarrow R_n \subseteq \{|z-z^*| < \delta\}$

Fix $\varepsilon > 0$. Choose $\delta > 0$ so small f is analytic on $\{|z-z^*|<\delta\}$ and $\left|\frac{f(z)-f(z^*)}{z-z^*} - f'(z^*)\right| < \varepsilon$ for $|z-z^*|<\delta$

ie., $|f(z) - f(z^*) - f'(z^*)(z-z^*)| < \varepsilon|z-z^*|$ for $|z-z^*| < \delta$.

Choose n so large $R_n \subseteq \{|z-z^*|<\delta\}$.

We know: $\int_{\partial R_n} dz = 0$ (b/c 1 is deriv. of z (anal.))

& $\int_{\partial R_n} z\,dz = 0$ (b/c z is deriv of $\frac{1}{2}z^2$ (anal.))

$$\eta(R_n) = \int_{\partial R_n} [f(z) - \underbrace{f(z^*)}_{\text{const}} - \underbrace{f'(z^*)(z-z^*)}_{\text{linear}}]\,dz$$

linear ← b/c $\int = 0$, don't change $\eta(R_n)$

$$|\eta(R_n)| \leq \int_{\partial R_n} \varepsilon|z-z^*|\,|dz|$$

← arc length measure

$|z-z^*| \leq d_n$ ← diagonal of R_n

Let L_n = perimeter of R_n

$|\eta(R_n)| \leq \varepsilon \cdot d_n \cdot L_n$

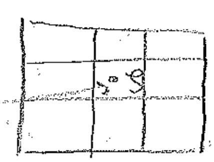
$d_n = 2^{-n} d$ (diagonal halved each time)
$L_n = 2^{-n} L$ (same)

$|\eta(R_n)| \leq 4^{-n} d L \varepsilon$
$|\eta(R)| \leq 4^n |\eta(R_n)| \leq d L \varepsilon \xrightarrow{\varepsilon \to 0} 0$
$\Rightarrow \eta(R) = 0.$

So, if $\partial_{\bar{z}} f = 0$, then $\int_\gamma f = 0$ for rectangles.
(soon $\Rightarrow f \in C^\infty$)

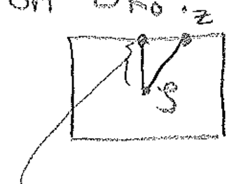
<u>Thm</u>: Let f be analytic on the region R' which is the rectangle R minus a finite # of pts, ζ_j.
If $\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0 \ \forall j$. Then $\int_{\partial R} f(z)\,dz = 0$

<u>Pf</u>: It suffices to consider the case of 1 exceptional ζ (else, divide R into subrects, each w/ 1 ζ).
Divide R into 9 rectangles.

R_0 

$\int_{\partial R} f\,dz = \int_{\partial R_0} f(z)$ (really sum of all subrects, but $\int$ of all other rects $= 0$)

R_0 can be small, ζ can be the center. We can choose R_0 so small $|f(z)| \leq \frac{\varepsilon}{z - \zeta}$ on ∂R_0.
$\left| \int_{\partial R} f\,dz \right| = \left| \int_{\partial R_0} f\,dz \right| \leq \varepsilon \int_{\partial R_0} \frac{|dz|}{|z - \zeta|}$.



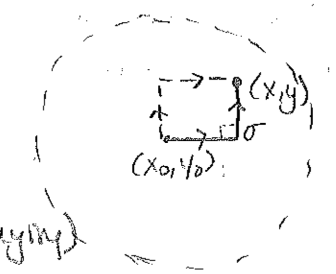
$\frac{1}{|z-\zeta|}$

$\leq 8\varepsilon \xrightarrow{\varepsilon \to 0} 0$ □

$|z - \zeta| \approx$ length of ∂R_0 w/ some const, 1/4

Thm: If f is analytic in a disk Δ & γ is a closed curve in Δ, then $\int_\gamma f\,dz = 0$.
[need $z(t)$ cts & $z'(t)$ exists for finite # of pts → piecewise smooth]

Pf: WTS $f\,dz$ is exact.



$F(x,y) := \int_\sigma f\,dz$

$\frac{\partial F}{\partial y} = i f(z)$ ($dz = dx + i\,dy$, but only y varying)

$\frac{\partial F}{\partial x} = f(z)$

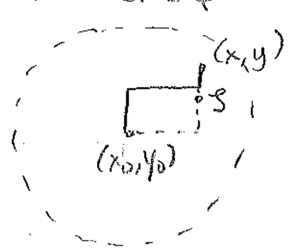
⇒ So F is analytic, $f = F'$, & $f\,dz$ is exact

⇒ $\int_\gamma f\,dz = 0$. $\forall$ closed curves γ.

- could choose dotted path by prev. thm.

Thm: Let f be analytic in the region Δ' which is obtained from the disk Δ by subtracting a finite # of pts ζ_j w/ $\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0$. Then $\int_\gamma f\,dz = 0$ $\forall$ closed curves γ in Δ'.

Pf: same except:



If ζ is on the path, choose a "staircase" path, & then same proof to show $\partial F/\partial y$, $\partial F/\partial x$ are correct.

Note: Replace disk w/ any open set that contains the inside of γ & $\int_\gamma f\,dz = 0$.

$\int_{\partial C} \frac{1}{z-a}\,dz = 2\pi i$ $\quad$ (a) C $\quad$ we'll see: $\int_\gamma \frac{1}{z-a}\,dz = 2\pi i n$, $n \in \mathbb{Z}$

n means something - winding #.

2/11 <u>Recall</u>: Cauchy's Thm: Let $f: \Delta \to \mathbb{C}$ be a holo. fcn on the open disk Δ & let γ be a piecewise smooth closed curve in Δ. Then $\int_\gamma f(z)\,dz = 0$.

Comment: This is even true if f is holomorphic except at a finite # of pts. ζ_j, so long as $\lim_{z \to \zeta_j} (z-\zeta_j) f(z) = 0$ (ie, it blows up not as badly as $1/z$).

Want to look at $\int_\gamma \frac{1}{z-\zeta}\,dz$

<u>Lemma</u>: If the piecewise smooth closed curve γ does not pass through $a \in \mathbb{C}$, then $\int_\gamma \frac{dz}{z-a}$ is an integer multiple of $2\pi i$.

<u>Pf</u>: Let γ be parameterized by $z(t)$, $\alpha \le t \le b$ s.t. $z'(t)$ exists & is cts at all but a finite # of pts.

Let $h(t) = \int_\alpha^t \frac{z'(t)}{z(t)-a}\,dt$

So $\frac{d}{dt} h(t) = \frac{z'(t)}{z(t)-a}$ where $z'(t)$ cts. Consider $\frac{d}{dt} e^{-h(t)}(z(t)-a) = 0$, at all but finite # of pts.

Conclude $e^{-h(t)}(z(t)-a) \equiv c$. Thus $e^{h(t)} = \frac{z(t)-a}{z(\alpha)-a}$ $(t=\alpha)$

$(= e^{-h(\alpha)}(z(\alpha)-a) \Rightarrow c = z(\alpha)-a)$, $e^{-h(\alpha)} = 1$ b/c $h(\alpha) = 0$

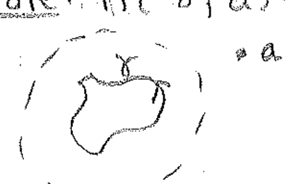
$z(\beta) = z(\alpha)$, so $e^{h(\beta)} = 1$

$\Rightarrow h(\beta) = 2\pi i n$ for some $n \in \mathbb{Z}$.

<u>Def</u>: In the setup as above, we define the <u>winding number</u> of γ about a, $n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} \in \mathbb{Z}$.

Also called the <u>index</u> of γ about a.

<u>Note</u>: $n(-\gamma, a) = -n(\gamma, a)$



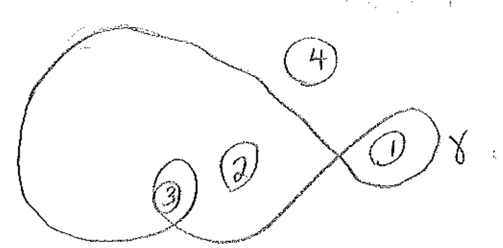
$n(\gamma, a) = 0$, b/c $= \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = 0$, by Cauchy's Thm

- γ as a pt. set, $z([\alpha,\beta])$, is cpt b/c image of cpt set under cts map.
- γ^c is open & is a union of ctd open components. In $\mathbb{C}\cup\{\infty\}$, only one of these components contains ∞. We call that the <u>unbounded component</u>.

$n(\gamma,a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$

$n(\gamma,\cdot): \mathbb{C}\setminus\gamma \to \mathbb{C}$. This is cts in a. (DCT)

Since $n(\gamma,\cdot): \mathbb{C}\setminus\gamma \to \mathbb{Z}$, & the only cts maps to $\mathbb{Z}$ are constant on each component, b/c $\mathbb{Z}$ has the discrete topology.



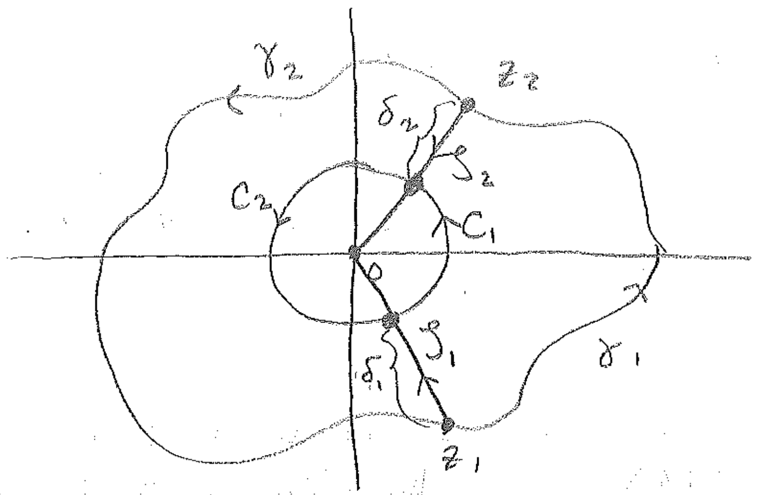
$n(\gamma,\cdot)$ constant on each of the 4 components.

- In ④, the unbdd component, take a so large that $\gamma \subseteq \{z \mid |z|<|a|\}$. Then $n(\gamma,a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = 0$ by Cauchy. ($\frac{1}{z-a}$ holo. on disk). $\Rightarrow n(\gamma,a)=0 \ \forall a\in$ ④.

Take $a=0$, b/c can just translate.

<u>Lemma</u>: Let z_1, z_2 be 2 pts on the piecewise smooth closed curve γ, which does not pass through 0. Denote the subarc of γ from z_1 to z_2 by γ_1 & the one from z_2 to z_1 by γ_2. Suppose z_1 lies in the lower half plane & z_2 lies in the upper half plane, & γ_1 not hit negative real axis & γ_2 not hit the positive real axis. (ie, γ goes around zero once) Then $n(\gamma,0)=1$.

Pf:



$\sigma_1 := \gamma_1 + \delta_2 - c_1 - \delta_1$ (closed loop on right)
$\sigma_2 := \gamma_2 + \delta_1 - c_2 - \delta_2$ (closed loop on left)
Cauchy: $n(\sigma_1, 0) = 0$ & $n(\sigma_2, 0) = 0$

$$n(\gamma,0) = n(c,0) + \underset{0}{\underset{\shortparallel}{n(\sigma_1,0)}} + \underset{0}{\underset{\shortparallel}{n(\sigma_2,0)}}$$

$$= n(c,0) = 1$$

□

Let $f(z)$ be holo on a disk, Δ. Let γ be a piecewise smooth closed curve in Δ. Fix $a \in \Delta$, $a \notin \gamma$.

$F(z) := \frac{f(z)-f(a)}{z-a}$. F holo except maybe at $z=a$.

Notice $\lim_{z\to a} F(z) = f'(a)$, so by Cauchy's thm, $\int_\gamma F(z)dz = 0$

ie, $\int_\gamma \frac{f(z)dz}{z-a} = f(a)\int_\gamma \frac{dz}{z-a} = f(a)\, n(\gamma,a)\cdot 2\pi i$

Formula: If $f:\Delta \to \mathbb{C}$, Δ an open disk, is holo & γ is a piecewise smooth closed curve not containing $a \in \mathbb{C}$, then

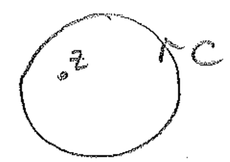
$$n(\gamma,a)\cdot f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a}.$$

- most common application is when $n(\gamma,a) = 1$.
  Then $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-a}$

Change notation:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad (\text{where } n(\gamma, z) = 1)$$

If we know f on a circle, C, then we know f inside of C: $f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$.



Consider $f: \Omega \to \mathbb{C}$, Ω any open set, analytic. Fix $a \in \Omega$, let $\Delta \subset \Omega$ be an open disk centered at a. (f holo. on Δ). Let C be a counterclockwise circle centered at a inside of Δ. We have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for } z \text{ inside of } C.$$

So, $f'(z) = \frac{d}{dz} f(z) = \frac{d}{dz} \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \xrightarrow{\text{next time}} \frac{1}{2\pi i} \int_C \frac{d}{dz} \frac{f(\zeta)}{\zeta - z}\, d\zeta$

(for z near a)

$$= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

. This is a holomorphic fcn & can diff:

$$f''(z) = \frac{d}{dz} f'(z) = \frac{1}{2\pi i} \int_C \frac{d}{dz} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta = \frac{2}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^n}.$$

If C is huge & z near center, $|f^{(n)}(z)| \leq \int_C \frac{|f(\zeta)|}{|\zeta - z|^n} |d\zeta| \cdot \frac{n!}{2\pi}$

(ie, can bound n^{th} deriv. by the fcn).

2/13 <u>Cauchy's Integral Formula</u>: Suppose $f:(\Delta \subseteq \Omega \text{ disk}) \to \mathbb{C}$ is analytic & γ is a piecewise smooth closed curve in Δ & if $a \in \Delta$, a not on γ, then

$$n(\gamma, a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$

- Usually, $n(\gamma, a) = 1$, then $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) d\zeta}{\zeta - z}$ $\forall z$.
- The same is true for $f: \Delta' \to \mathbb{C}$ where Δ' is Δ minus a finite # of pts, ζ_j, & $\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0$

<u>Lemma</u>: Suppose $\Phi(\zeta)$ is cts on the arc γ, the fcn $F_n(z) = \int_\gamma \frac{\Phi(\zeta)}{(\zeta - z)^n} d\zeta$ in the regions determined by γ. is analytic & $F_n'(z) = n F_{n+1}(z)$

(ie, can pull $\frac{d}{dz}$ inside the integral)

<u>Pf</u>: 1^{st}: $F_n(z)$ is cts: Let $\zeta(t)$, $a \le t \le b$, parameterize γ.

$$F_n(z) = \int_a^b \frac{\Phi(\zeta(t))}{(\zeta(t) - z)^n} \zeta'(t)\, dt$$

$$\lim_{z \to z_0} F_n(z) = \lim_{z \to z_0} \int_a^b \frac{\Phi(\zeta(t))}{(\zeta(t) - z)^n} \zeta'(t)\, dt$$

$$\overset{DCT}{=} \int_a^b \frac{\Phi(\zeta(t))}{(\zeta(t) - z_0)^n} \zeta'(t)\, dt = F_n(z_0).$$

(DCT b/c $\Phi(\zeta(t))$ cts & $\zeta(t)$ bdd $\Rightarrow \Phi(\zeta(t))$ bdd, & $(\zeta(t) - z)$ bdd b/c inside curve

By induction: $n=1$

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_\gamma \Phi(\zeta) \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right] d\zeta$$

$$= \frac{1}{\cancel{z - z_0}} \int_\gamma \Phi(\zeta) \frac{\cancel{z - z_0}}{(\zeta - z)(\zeta - z_0)}\, d\zeta$$

$$\lim_{z \to z_0} \frac{F_1(z) - F_1(z_0)}{z - z_0} = \lim_{z \to z_0} \int_\gamma \frac{\Phi(\zeta) \frac{1}{\zeta - z_0}}{\zeta - z}\, d\zeta \overset{\text{cts}}{=} \int_\gamma \frac{\Phi(\zeta)}{(\zeta - z_0)^2}\, d\zeta$$

Suppose $F_{n-1}'(z) = (n-1) F_n(z)$.

$$F_n(z) - F_n(z_0) = \int_\gamma \Phi(\zeta) \left[\frac{1}{(\zeta - z)^n} - \frac{1}{(\zeta - z_0)^n} \right] d\zeta$$

$$= \int_\gamma \Phi(\zeta) \left[\frac{1}{(\zeta - z)^{n-1}(\zeta - z_0)} - \frac{1}{(\zeta - z_0)^n} + \frac{1}{(\zeta - z)^n} - \frac{1}{(\zeta - z)^{n-1}(\zeta - z_0)} \right] d\zeta$$

$$= \int_\gamma \phi(\zeta)\left[\frac{1}{(\zeta-z)^{n-1}(\zeta-z_0)} - \frac{1}{(\zeta-z_0)^n}\right]d\zeta + \int_\gamma \phi(\zeta)\left[\frac{z-z_0}{(\zeta-z)^n(\zeta-z_0)}\right]d\zeta$$

$$\lim_{z\to z_0} \frac{F_n(z)-F_n(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{1}{z-z_0}\underbrace{\left[\int_\gamma \frac{\phi(\zeta)\frac{1}{\zeta-z_0}}{(\zeta-z)^{n-1}}d\zeta - \int_\gamma \frac{\phi(\zeta)\frac{1}{\zeta-z_0}}{(\zeta-z_0)^{n-1}}d\zeta\right]}_{=(n-1)F_{n+1}(z_0) \text{ by ind. hyp.}}$$

$$+ \lim_{z\to z_0}\int_\gamma \frac{\phi(\zeta)\frac{1}{\zeta-z_0}}{(\zeta-z)^n}d\zeta$$

$$= (n-1)F_{n+1}(z_0) + F_{n+1}(z_0)$$

$$= n F_{n+1}(z_0) \qquad \square$$

<u>Conclusion</u>: $f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta$, when $n(\gamma, z) = 1$.

We can conclude that the derivative of an analytic fcn is analytic:

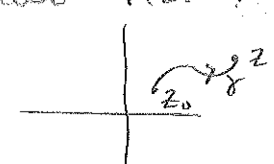
Let $f: (\Omega \subseteq \mathbb{C} \text{ open}) \to \mathbb{C}$ be analytic. Fix $z_0 \in \Omega$.
Let B be a disk centered at z_0 in Ω & C be a circle centered at z_0 in B. Inside of C, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)}$$

Thus $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}$. This is analytic, so $f'(z)$ is analytic inside of C. But $z_0 \in \Omega$ was arbitrary, so $f'(z)$ is analytic in Ω.

<u>Morera's Thm</u>: If $f: \Omega \to \mathbb{C}$ cts and $\int_\gamma f(z)dz = 0$ $\forall$ closed curve γ, then f is analytic.

<u>Pf</u>: In this case, we saw that $f(z) = F'(z)$ where F was analytic. So f is analytic.



$F(z) = \int_\gamma f(\zeta)d\zeta$

- $\forall \gamma$ closed curve, $\int_\gamma f(z)dz = C$. C independent of curve, so take constant curve, & so $C = 0$. (that's why we need $\int_\gamma = 0$).

analytic fcn defined on whole complex plane

Liouville's Thm: A fcn $f: \mathbb{C} \to \mathbb{C}$ which is analytic (f is called entire) which is bdd (ie $|f(z)| \le M$) is constant.

Pf: Fix $a \in \mathbb{C}$. Let C be the circle of radius R centered at a.

$$|f'(a)| = \left| \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - a)^2} d\zeta \right|$$

$$\le \frac{1}{2\pi} \int_C \frac{|f(\zeta)|}{\underbrace{|\zeta - a|^2}_{=R \text{ b/c on circle w/ center } a}} |d\zeta|$$

$$\le \frac{1}{2\pi} \frac{M}{R^2} \cdot \underbrace{2\pi R}_{\text{Length of } C}$$

$$\xrightarrow{R \to \infty} 0$$

We conclude $f'(z) \equiv 0$ (since a was arbitrary), so f = constant.

Fundamental Thm of Algebra: Let $P(z) = \sum_{j=0}^{n} a_j z^j$, $a_n \ne 0$, $a_j \in \mathbb{C}$, $n > 0$. Then $\exists z_0$ w/ $P(z_0) = 0$.

(every polynomial over the complexes has a root)

Pf: Suppose $P(z) \ne 0$ $\forall z$. Let $f(z) = \frac{1}{P(z)}$, so f is entire (b/c P(z) is entire & $\ne 0$).

$$|P(z)| \ge \underbrace{|a_n z^n|}_{|a_n||z|^n} - \sum_{j=0}^{n-1} |a_j z^j| \xrightarrow{z \to \infty} \infty$$

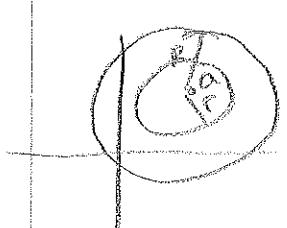
for n large, $|z|^n$ dominates all $|z|^j$

$\Rightarrow |f(z)| \to 0$ as $z \to \infty$

$\Rightarrow$ f bdd

Thus f(z) is constant $\Rightarrow$ P(z) is constant, ↯.

Let f be analytic on a disk of radius R centered at a & let $r < R$. Let $M_r = \sup_{\{z \in \text{circle of radius } r\} C_r} |f(z)|$ ($\exists$ b/c f cts & C closed & bdd)

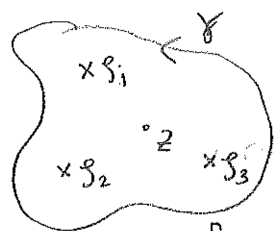


$$|f^{(n)}(a)| = \left| \frac{n!}{2\pi i} \int_{C_r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right|$$

$$\le \frac{n!}{2\pi} \int_{C_r} \frac{M_r}{r^{n+1}} |d\zeta| = \frac{n!}{\cancel{2\pi}} \cdot \frac{M_r}{r^{n+1}} \cdot \cancel{2\pi r}$$

$$= \frac{n! M_r}{}$$

• So derivatives go to ∞ as $n \to \infty$, but they can't grow arbitrarily fast.



$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi$$

Assume $\lim_{z \to \xi_j} (z - \xi_j) f(z) = 0$

Can reconstruct f $\forall z$ in γ, $z \neq \xi_j$, by knowing f on γ.

2/15 p.120/1,2 ; p.123/2,3,4,5

Thm: Suppose $f(z)$ analytic Ω' obtained from Ω by removing a pt a. Then a necessary & sufficient condition that there exists an analytic fcn on Ω which agrees w/ f on Ω' is that:
$\lim_{z \to a} (z-a) f(z) = 0$.
The extension is unique

* f has a singularity at a, if f is holo. everywhere but a.
* If $\lim_{z \to a} (z-a) f(z) = 0$, then a is a removable singularity

Pf: Necessity & uniqueness are trivial: the extended fcn must satisfy $\lim_{z \to a} f(z) = f(a)$ (b/c cts)
For sufficiency, let $C \subseteq \Omega$ be a circle centered at a, & whose inside is in Ω. Cauchy's formula yields for z inside of C, $z \neq a$,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z}\, d\xi .$$

The r.h.s. gives an analytic fcn inside of C

The desired extended fcn is $\begin{cases} f(z), & z \neq a \\ \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - a}\,d\zeta, & z = a \end{cases}$

<u>Ex</u>: $\frac{1}{(z-a)^n}$ does <u>not</u> have a removable singularity.

Ω is connected & open; $f : \Omega \to \mathbb{C}$ is analytic. Fix $a \in \Omega$.

Define $F(z) = \frac{f(z) - f(a)}{z - a}$. For $z \neq a$, F is holomorphic.

$\lim_{z \to a} F(z) = f'(a)$, so this is a removable singularity.

Let $f_1(z) = \begin{cases} F(z), & z \neq a \\ f'(z), & z = a \end{cases}$. f_1 is holo. on Ω.

Set $f_2(z) = \begin{cases} \frac{f_1(z) - f_1(a)}{z - a} \\ f_1'(a), & z = a \end{cases}$ holo. on Ω, Continue ... $\leftarrow$ (abuse notation & don't write this)

$f(z) = f(a) + (z-a) f_1(z)$

$f_1(z) = f_1(a) + (z-a) f_2(z)$

$\vdots$

$f_n(z) = f_n(a) + (z-a) f_{n+1}(z)$

So, $f(z) = f(a) + (z-a) f_1(a) + (z-a)^2 f_2(a) + (z-a)^3 f_3(a) + \cdots + (z-a)^n f_n(z)$.

$f^{(n)}(a) = n!\, f_n(a)$

<u>Thm</u>: If $f(z)$ is analytic in Ω containing a, we can write

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1}$$

$$+ f_n(z)(z-a)^n, \quad \text{where } f_n(z) \text{ holo. on } \Omega.$$

* 1st n terms look like Taylor series, & error term is holo. fcn.

$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)\,d\zeta}{\zeta - z}$, C a circle centered at a.

If solve for $f_n(z)$, get one term involving $f(z)$. Rest of terms are constant times $F_\nu(a) = \int_C \frac{d\zeta}{(\zeta-a)^\nu(\zeta-z)}$, $\nu \geq 1$.

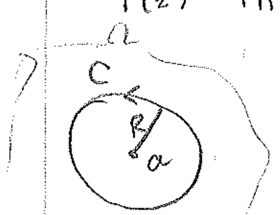
$$F_1(a) = \int_C \frac{d\zeta}{(\zeta-a)(\zeta-z)} = \frac{1}{z-a}\int_C\left(\frac{1}{\zeta-z} - \frac{1}{\zeta-a}\right)d\zeta = \frac{1}{z-a}(2\pi i - 2\pi i) = 0$$

partial frac. decomp.

$F_{\nu+1}(a) = \frac{F_1^{(\nu)}(a)}{\nu!}$, so $F_\nu \equiv 0\ \forall \nu$. $\Rightarrow f_n(\zeta) = \frac{f(\zeta)}{(\zeta-a)^n}$.

We conclude $f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-a)^n(\zeta-z)}\,d\zeta$

Suppose $f(a)=0$, $f^{(\nu)}(a)=0\ \forall \nu$. (ie, f vanishes to ∞ order).
We will see $f \equiv 0$.
To see this, we have $\forall n$
$f(z) = f_n(z)(z-a)^n$ (b/c 1^{st} n terms vanish)



Let $M = \sup_{z\in C} |f(z)|$

$$|f_n(z)| = \left|\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta-a)^n(\zeta-z)}\right|$$

$$\leq \frac{1}{\cancel{2\pi}}\cdot\frac{M}{R^{n-1}(R-|z-a|)}\cdot \cancel{2\pi R} = \frac{M}{R^{n-1}(R-|z-a|)}$$

for $|z-a|<R$.

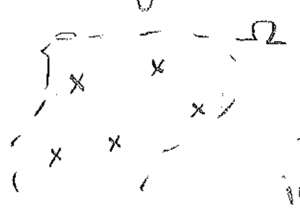
$$|f(z)| \leq \left(\frac{|z-a|}{R}\right)^n\cdot\frac{MR}{R-|z-a|} \to 0 \text{ as } n\to\infty.$$

So, $f\equiv 0$ inside of C.
Ω is a connected open set.
Let E_1 = the set on which $f(z)$ & all derivatives vanish.
E_2 = the set of pts where some derivative is nonzero.
$E_1 \sqcup E_2 = \Omega$.
E_1 is open, as shown above. E_2 is open by continuity of derivatives. (ie, If $f^{(n)}(z)=0$, $\exists$ open disk around z s.t. $f^{(n)}=0$ b/c cts)

If 2 open disjoint sets union to an open ctd set, one must be empty.
Either $E_1 = \Omega \Rightarrow f \equiv 0$ or $E_2 = \Omega \Rightarrow f$ never vanishes to ∞-order.

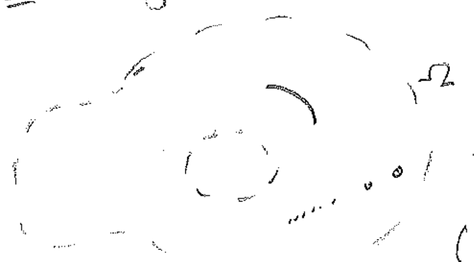
Suppose $f \not\equiv 0$. Then $\exists$ a first derivative $f^{(h)}(a)$ which is different from zero. We say f has a <u>zero of order h</u> at a.
(proved every zero has finite order)

$f(z) = (z-a)^h f_h(z)$, $f_h(z)$ analytic & $f_h(a) \neq 0$, else would have zero of higher order.
(like polys, can factor out zeros)
$f_h(a) \neq 0$ on a nbhd of a, so $f(z) = 0$ on this nbhd iff $z = a$.
The zeros of an analytic fcn are <u>isolated</u>.



If the zeros accumulate to a pt, that pt must be on boundary of Ω.
ie, there are no limit pts of zeros.

<u>Thm</u>: If f, g are holo. fcns on Ω & $f = g$ on a set with a limit pt in Ω, then $f \equiv g$.
<u>Pf</u> $f - g = 0$ on a set w/ a limit pt, so $f - g \equiv 0$.



If know f on), know f everywhere. Could know f on countable seq. w/ limit.
(finite not enough, limit on $\partial\Omega$ not enough, but anything else is)

Suppose $f(z)$ is analytic on a nbhd of a except possibly at a. $0<|z-a|<\delta$.
We call a an <u>isolated singularity</u> of f.

① <u>Removable sing</u> : $\lim_{z\to a}(z-a)f(z)=0$

• f extends to a

② $\lim_{z\to a} f(z)=\infty$. we will see $f(z)=\frac{a_n}{(z-a)^n}+\frac{a_{n-1}}{(z-a)^{n-1}}+\cdots$
$a_1(z-a)+\phi(z)$
holo.

<u>Pole</u>

③ <u>Essential Singularities</u>
- on any nbhd, f takes almost all values.

2/18 Let f be analytic on a nbhd of a pt $a\in\mathbb{C}$, except possibly not at a. f is analytic on $0<|z-a|<\delta$.
We call a an <u>isolated singularity</u> of f.

① If $\lim_{z\to a}(z-a)f(z)=0$, f extends to an analytic fcn at a w/ $f(a)=\lim_{z\to a} f(z)$.

② If $\lim_{z\to a} f(z)=\infty$ (ie, $\lim_{z\to a}|f(z)|=\infty$ ← def). We call

a a <u>pole</u> of f. Set $f(a)=\infty$.
$\exists\, \delta'>0$ s.t. $f(z)\neq 0$ on $0<|z-a|<\delta'$. Set $g(z)=\frac{1}{f(z)}$.
g is analytic on $0<|z-a|<\delta'$ and
$\lim_{z\to a} g(z)=\lim_{z\to a}\frac{1}{f(z)}=0$. (case 1)

So, g has a removable singularity at $z=a$ with $g(a)=0$. $g\not\equiv 0$, so a is a zero of finite order h.
$\Rightarrow g(z)=(z-a)^h g_h(z)$, $g_h(a)\neq 0$
h is called the <u>order</u> of the pole of f at a.
On $0<|z-a|<\delta'$,
$f(z)=(z-a)^{-h}\frac{1}{g_h(z)}$.

Def: A function f which is analytic in a region Ω except for poles is called meromorphic, ie $\forall a \in \Omega$ $\exists$ a nbhd $|z-a|<\delta$ s.t. either f is analytic on $|z-a|<\delta$ or f is analytic on $0<|z-a|<\delta$ & $\lim_{z \to a} f(z) = \infty$.

(zeros are always isolated, poles are isolated by def.)

Let f, g be analytic on Ω, $g \not\equiv 0$. Then f/g is meromorphic.
More generally, if f, g are meromorphic, so are $f+g$, fg, f/g if $g \not\equiv 0$. (field)

Consider 2 conditions. $\alpha \in \mathbb{R}$. (f has an iso. sing at $z=a$)
(1): $\lim_{z \to a} |z-a|^\alpha |f(z)| = 0$

(2) $\lim_{z \to a} |z-a|^\alpha |f(z)| = \infty$

Suppose (1) holds for some α. Then (1) holds for some integer $m \geq \alpha$ (b/c then must hold for all bigger α).
So $(z-a)^m f(z)$ has a removable sing. at a.
Either $f \equiv 0$ or $(z-a)^m f(z)$ has a zero of finite order k. Then

$$(z-a)^m f(z) = (z-a)^k g(z) \quad , g \text{ analytic everywhere } \& \ g(a) \neq 0$$

In this case, (1) holds for all $\alpha > h = m-k \in \mathbb{Z}$ and (2) holds for all $\alpha < h$.

Suppose (2) holds for some α. Then it holds for some integer $n < \alpha$. $(z-a)^n f(z)$ has a pole at a of order $l \in \mathbb{N}$. So

$$(z-a)^n f(z) = (z-a)^{-l} g(z), \quad g(a) \neq 0.$$

Set $h = n+l$, & (1) holds $\forall \alpha > h$ & (2) holds $\forall \alpha < h$.

3 possiblities:

① (1) holds $\forall \alpha$ & $f \equiv 0$.

② $\exists$ an integer h s.t. (1) holds $\forall \alpha > h$ & (2) holds $\forall \alpha < h$. ($h<0 \Rightarrow a$ a pole, $h>0 \Rightarrow a$ a zero, $h=0 \Rightarrow$ nonzero but analytic)

③ Neither (1) nor (2) holds for any α.

In the 2nd case, f is algebraic of order h at a.

Suppose $f(z)$ has a pole of order h at a.
(ie, h is minimal s.t. $(z-a)^h f(z)$ has a removable singularity at $z=a$). Taylor Series:

$$(z-a)^h f(z) = B_h + B_{h-1}(z-a) + B_{h-2}(z-a)^2 + \cdots + B_1(z-a)^{h-1} + \phi(z)(z-a)^h,$$ ϕ analytic, $B_h \neq 0$ b/c h minimal.

$\Rightarrow$ for $z \neq a$,

$$f(z) = \underbrace{B_h(z-a)^{-h} + B_{h-1}(z-a)^{-h+1} + \cdots + B_1(z-a)^{-1}}_{\text{singular part of } f(z) \text{ at } a} + \phi(z)$$

$\rightarrow$ 2 fcns w/ same singular part at a differ by analytic fcns.

In the 3rd case, neither (1) nor (2) holds for any α.
We call a an essential singularity of f.

Thm: An analytic fcn comes arbitrarily close to any complex value in every nbhd of an essential singularity.
(every nbhd of a mapped to a dense set of $\mathbb{C}$)

Pf: Suppose not. Then $\exists A \in \mathbb{C}, \delta > 0$ s.t.
$|f(z)-A| > \delta$ in a nbhd of a.
For $\alpha < 0$, $\lim_{z \to a} |z-a|^{\alpha} \underbrace{|f(z)-A|}_{b/c > \delta} = \infty$

So a is not an essential singularity of $f(z)-A$.
Thus $\exists \beta$ w/ $\lim_{z\to a} |z-a|^{\beta}|f(z)-A| = 0$. We can assume $\beta > 0$. But $\lim_{z \to a} |z-a|^{\beta}|A| = 0$

$\Rightarrow \lim_{z\to a} |z-a|^{\beta}|f(z)| = 0 \Rightarrow a$ not an essential singularity ↯.

Ex: $e^{1/z}$ has an essential sing. at $z=0$.

On $\mathbb{C} \cup \{\infty\}$, we say ∞ is an isolated sing. of $f(z)$ if f is analytic on $|z|>R$ for some R.
(ie, punctured disks centered at ∞)
Set $g(z) = f(1/z)$. g has an iso. sing. at 0. By convention we say the sing. of f at ∞ is removable, a pole, or essential if the same holds for g at 0.

Suppose f is algebraic at ∞.
$\lim_{z\to\infty} z^{-n} f(z)$ is neither 0 nor ∞ for some n.

If $n > 0$, f has a pole at ∞. The sing. part of f at ∞ is a poly. of order n.

<u>Counting Zeros</u>

Ex: $g \neq 0$. Take $f(z) = (z-z_0)g(z)$. $z_0 \in C$, a circle
$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz$ (f'/f deriv. of logarithm)

$= \frac{1}{2\pi i}\int_C \underbrace{\frac{1}{(z-z_0)}}_{=2\pi i} + \underbrace{\frac{g'}{g}}_{\int = 0 \text{ by Cauchy's Thm}}\,dz = 1$, & $1 = \#$ of 0's in C.

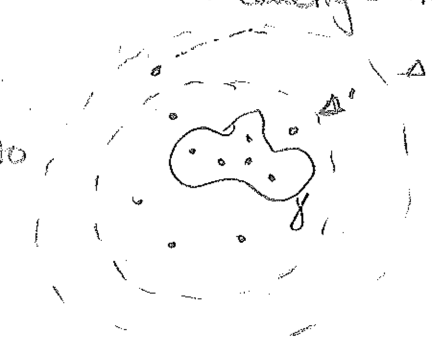
2/20 Let Δ be a disk, $f: \Delta \to \mathbb{C}$ analytic, $(f \not\equiv 0)$ γ closed curve in Δ that does not pass through a zero of f.
Replace Δ w/ a smaller disk $\Delta' \subseteq \Delta$ w/ $\gamma \subseteq \Delta'$, & since Δ' is smaller, f can only have finitely many zeros on Δ'. If it had ∞-ly many, they'd have a limit pt in $\overline{\Delta'} \subseteq \Delta$, which is cpt, & so $f \equiv 0$.
List the zeros: $z_1, \ldots, z_n$, counted w/ multiplicity, ie, if f has a 0 of order k at z_0, z_0 appears k times.
$f(z) = (z-z_1)\cdots(z-z_n)\,g(z)$, $g(z) \neq 0$, so $\frac{1}{g(z)}$ holo & $g'(z)$ holo.

$$\frac{f'(z)}{f(z)} = \text{(logarithmic derivative of } f\text{)}$$
$$= \frac{1}{z-z_1} + \frac{1}{z-z_2} + \cdots + \frac{1}{z-z_n} + \frac{g'(z)}{g(z)}$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{n} \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_j}\,dz + \underbrace{\int_\gamma \frac{g'(z)}{g(z)}\,dz}_{=0}$$
$$= \sum_{j=1}^{n} n(\gamma, z_j)$$

0 b/c $\frac{g'}{g}$ holo $\Rightarrow$ Cauchy's Thm.

- any z_j not in γ adds 0 to the sum.



<u>Thm</u>: Let z_j be the zeros of f, which is analytic in a disk Δ, zeros listed w/ multiplicities, & γ does not pass through a zero. Then
$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}.$$

LHS has only finitely many nonzero terms.

$w = f(z)$, $\gamma \subseteq \Delta \rightsquigarrow f(\gamma) = \Gamma$ a closed curve in w-plane

$$\underbrace{\frac{1}{2\pi i} \int_\Gamma \frac{dw}{w}}_{} = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\, dz$$

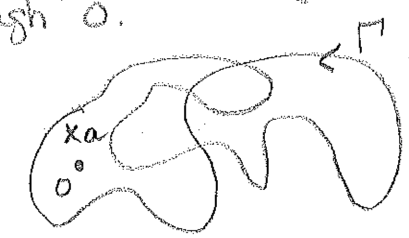
$$n(\Gamma, 0) = \sum_j n(\gamma, z_j)$$

- If γ were a circle, then $n(\gamma, z_j) = 0$ or 1.
  $\sum n(\gamma, z_j) = \#$ of zeros of f inside of γ, counted w/ multiplicity.

  $g(z) = f(z) - a$, $a \in \mathbb{C}$, $g(z) \neq 0$ on γ (ie., $f(z) \neq a$ on γ)
  - Zeros of $g(z)$ are solutions to $f(z) = a$. Call them $z_j(a)$ (w/ mult.)

  $$\Rightarrow \sum_j z_j(a) = \frac{1}{2\pi i}\int_\gamma \frac{g'}{g} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a}\, dz = n(\Gamma, a)$$

- γ doesn't pass through a zero of $f \Leftrightarrow \Gamma$ doesn't pass through 0.



$n(\Gamma, 0) = \#$ of solutions of $f = 0$.
$n(\Gamma, a) = \#$ of sol'ns to $f = a$
if a close to 0 (& in same component of Γ).

- If a & b are in the same region determined by Γ, so
  $n(\Gamma, a) = n(\Gamma, b) \Rightarrow \sum_j n(\gamma, z_j(a)) = \sum_j n(\gamma, z_j(b))$

<u>Thm</u>: Suppose f analytic & non-constant near $z_0 \in \mathbb{C}$, & $f(z_0) = w_0$. Suppose $f(z) - w_0$ has a zero of order n at z_0. If $\varepsilon > 0$ is small enough, $\exists$ $\delta > 0$ s.t. $|a - w_0| < \delta \Rightarrow f(z) = a$ has n roots in the disk $|z - z_0| < \delta$

Pf: We can choose $\varepsilon > 0$ s.t. $f(z)$ is defined & analytic for $|z-z_0| \leq \varepsilon$, & z_0 is the only zero of $f(z)-w_0$ in the disk (b/c zeros are isolated).
Let γ be the circle $|z-z_0| < \varepsilon$, & $\Gamma = f(\gamma)$. $w_0 \in \mathbb{C} \setminus \Gamma$.
So $\exists$ nbhd $|w-w_0| < \delta$ which does not intersect Γ (b/c Γ^c is open). All values of a in $|w-w_0| < \delta$ have same winding #, $(n(\Gamma, a)) = \#$ of sol'ns of $f(z)=a$.
$n(\Gamma, w_0) = n$

• If $\varepsilon > 0$ is small enough, the roots of $f(z)=a$ are simple (ie, $f(z)-a=0$ has zero of order 1 at each root) b/c take $\varepsilon > 0$ s.t. $f'(z)$ vanishes at only z_0 on $|z-z_0| < \varepsilon$. (f' analytic so its zeros isolated).

Cor: Analytic fcns take open sets to open sets.

$\{w \mid |w-w_0| < \delta\} \subseteq \{f(z) \mid |z-z_0| < \varepsilon\}$
↑ pt in image, & ball around that pt is in image.

Cor: If $n=1$, i.e. $f(z_0)=w_0$ & $f'(z_0) \neq 0$, then there is a 1-1 corresp. btwn
$\{w \mid |w-w_0| < \delta\}$ & an open subset of $\Delta = \{z \mid |z-z_0| < \varepsilon\}$
↑ every pt here hit exactly once by pts in here ↗
$f: f^{-1}(\{w \mid |w-w_0| < \delta\}) \to \{w \mid |w-w_0| < \delta\}$ is a bijection, cts w/ cts image, so this is a homeomorphism. $f^{-1}(w)$ is analytic.

Thm (Maximum Principle): If f analytic & non-constant on an open set Ω, then $|f(z)|$ does not attain a maximum.

Pf: If $w_0 = f(z_0)$ is a value taken in Ω, then $\exists$ nbhd $\{|w-w_0|<\varepsilon\}$ also in the image of Ω. There is a pt in this nbhd w/ modulus $> w_0$ (ie, take smth further away). $\square$

Cor: If $f(z)$ is cts on a closed & bdd (ie cpt) set E & f is analytic on the interior of E, then the max of $|f(z)|$ is attained on ∂E

(b/c cts fcn, $|f(z)|$, attains max on cpt set)

f need only be cts on ∂E, but analytic inside.

Let's suppose f analytic on $|z|<1$ & cts on $|z|\le 1$. If $|f(z)|\le M$ for $|z|=1$, then $|f(z)|\lneq M$ for $|z|<1$, with equality if f is constant.

Schwartz's Lemma: If $f(z)$ is analytic on $|z|<1$ & $|f(z)|\le 1$ & $f(0)=0$, then $|f(z)|\le|z|$ & $|f'(0)|\le 1$. If $|f(z)|=|z|$ for some $z\ne 0$, then $f(z)=cz$, with $|c|=1$.

Pf: Apply max. principle to $f_1(z)=\begin{cases} f(z)/z, & z\ne 0 \\ f'(0), & z=0\end{cases}$

on $|z|=r<1$. $|f_1(z)|\le\frac{1}{r}$ on $|z|=r$ (b/c $|f(z)|\le 1$), so $|f_1(z)|<1/r$ on $|z|<r$. Take $r\to 1$. Then $|f_1(z)|\le 1 \Rightarrow |f(z)|\le|z|$.

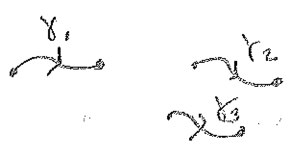
equality only if $f_1(z)=c \Rightarrow f(z)=cz$.

2/22 p.129/2,4,5 ; p.136/1,5 ; p.148/4

<u>Thm</u>: If $f(z)$ is analytic on $|z|<1$ & $|f(z)|\le 1$, $f(0)=0$, then $|f(z)|\le|z|$ & $|f'(0)|\le 1$. If $|f(z)|=|z|$ for some $z\ne 0$ or $|f'(0)|=1$, then $f(z)=cz$, $|c|=1$.

- If $|f(z)|\le M$, can use $g(z)=f(z)/M$, & then $|g(z)|\le 1$, etc.
- If $|z|<R$, use $g(z)=f(Rz)$ & then $|z|<1$ for g.
- If have ½ plane, use a linear frac. transf to map it to a disk, then apply thm.

$$\int_\gamma f\,dz = \sum \int_{\gamma_j} f\,dz$$

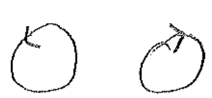


Let $\gamma=\gamma_1+\gamma_2+\gamma_3$, & then
$$\int_\gamma f\,dz := \sum \int_{\gamma_j} f\,dz$$

Let $\gamma = a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n$; & then $\int_\gamma f\,dz := \sum a_j \int_{\gamma_j} f\,dz$
(γ is a formal $\mathbb{Z}$-linear combination of γ_j's)

We call such γ a <u>chain</u>.
If γ_1 & γ_2 are chains, we say $\gamma_1=\gamma_2$ if $\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$ $\forall f$ (not nec. analytic).

<u>Def</u>: A chain is called a <u>cycle</u> if it can be represented as a sum of closed curves.



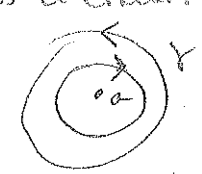
- if $p\,dx+q\,dy$ is exact, then $\int_\gamma p\,dx+q\,dy=0$ $\forall$ cycles γ.

Def: A region is said to be <u>simply connected</u> if its complement wrt $\mathbb{C} \cup \{\infty\}$ is connected.

Ex: ①  annulus not s. ctd.

②  disk is s. ctd. ③ half-plane is s. ctd.

If γ is a chain, we define $n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$.

 γ $n(\gamma,a)=0$ (they have w. #'s 1 & -1)

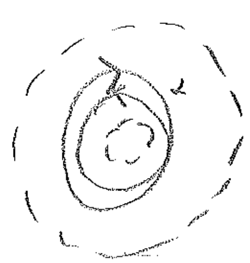
Thm: A region Ω is s. ctd iff $n(\gamma,a)=0$, $a \notin \Omega$, γ any cycle in γ. (We take this as a fact).

Ex:

 $n(\gamma,a)=1$  $\cdot a$ $n(\gamma,a)=0$

Def: A cycle γ in a region Ω is said to be <u>homologous to zero</u> if $n(\gamma,a)=0$ $\forall a \notin \Omega$. γ_1 is said to be homologous to γ_2 if $\gamma_1 - \gamma_2$ is homologous to 0.

γ is homo. to 0
δ is not homo. to 0

 cycle α is homo. to 0

In a s. ctd region, every cycle is homologous to 0.
(can use this as a def. of s. ctd)

Thm (Cauchy): If $f(z)$ is analytic in a region Ω, then $\int_\gamma f dz = 0$ for every cycle which is homo. to 0.

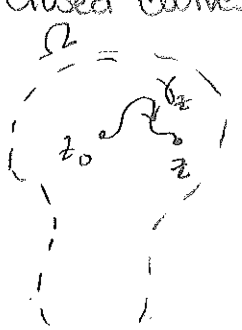
Cor: If $f(z)$ is analytic in a s. ctd Ω, then $\int_\gamma f dz = 0$ $\forall$ cycles in Ω.

Cor:
If f is analytic in Ω & γ_1 & γ_2 are 2 cycles in Ω w/ γ_1 homo. to γ_2, then $\int_{\gamma_1} f dz = \int_{\gamma_2} f dz$.
Pf: $\int_{\gamma_1 - \gamma_2} f\, dz = 0$ by Cauchy's Thm.

Ex:

Want $\int_{\gamma_1} f dz$. Note, γ_1 is homo. to a circle, so we can compute $\int_{\gamma_2} f dz = \int_{\gamma_1} f dz$.

Suppose f is analytic on a s. ctd region Ω. Then $\forall$ closed curves γ, $\int_\gamma f dz = 0$.



$F(z) := \int_{\gamma_z} f(\zeta) d\zeta$. This is well defined by Cauchy's Thm.
We showed $F' = f$.

- On a simply ctd region every analytic fcn has an analytic antideriv.
- $1/z$ does not have an antideriv. on $\mathbb{C} \setminus \{0\}$. It does on $\mathbb{C} \setminus \{\text{neg. reals}\}$, b/c that's s. ctd.

__Cor__: If $f(z)$ is analytic & never zero on a s.ctd region Ω, then it is possible to define an analytic logarithm of $f(z)$ & an analytic $\sqrt[n]{f(z)}$.

__Pf__: $\exists$ an analytic fcn $F(z)$ w/ $F'(z) = \frac{f'(z)}{f(z)}$.

Then $f(z)e^{-F(z)}$ has deriv. 0. Let $z_0 \in \Omega$, pick a choice of $\log f(z_0)$.

$e^{F(z) - F(z_0) + \log f(z_0)} = f(z)$ (plug in $z = z_0$) & $f(z)e^{-F(z)}$ is const, so const. must be 1.

So set $\log f(z) = F(z) - F(z_0) + \log f(z_0)$

$\sqrt[n]{f(z)}$ can be $e^{1/n \log f(z)}$.

__Ex__: Ω is s.ctd & f is analytic on Ω, f never 0. Write $f(z) = e^{g(z)}$ where g analytic on Ω (try this if you have no ideas...)

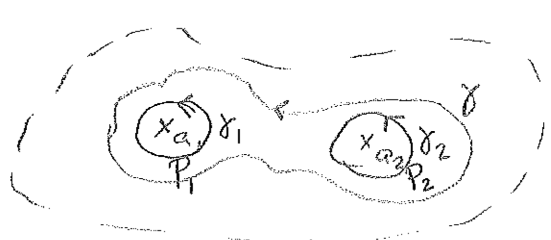
__Note__: Cauchy's integral formula works for cycles w/ same proof:

$n(\gamma, a) f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz.$

Suppose $f(z)$ analytic in Ω w/ a finite # of singularities $a_1, \dots, a_n$. $0 < |z - a_1| < \delta$

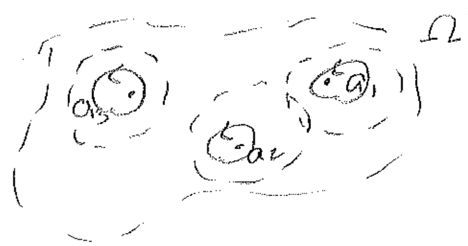
$P_1 := \int_{C_1} f(z)\,dz$. set $R_1 = \frac{P_1}{2\pi i}$

$\int_{C_1} f(z) - \frac{R_1}{z - a_1}\,dz = P_1 - P_1 = 0.$



so $\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2}$

2/25 Let f be analytic in Ω except for a finite number of singularities $a_1, \dots, a_n \in \Omega$, & $\Omega' = \Omega \setminus \{a_1, \dots, a_n\}$, so f anal. on Ω'. For each a_j $\exists$ $\delta_j > 0$, s.t. $\{0 < |z - a_j| < \delta_j\} \subseteq \Omega'$. Let C_j be a circle w/ center a_j & radius $< \delta_j$. Set $P_j = \int_{C_j} f\,dz$, & $R_j = P_j / 2\pi i$



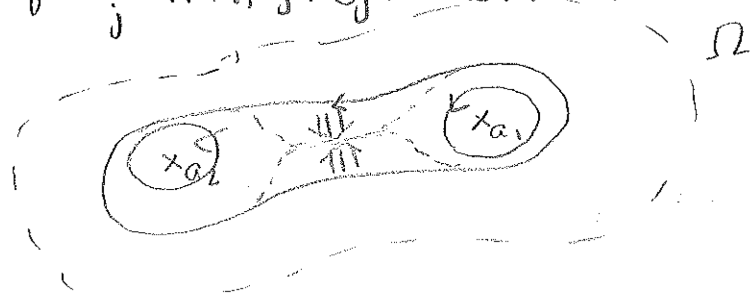
$$\int_{C_j} f(z) - \frac{R_j}{z - a_j}\,dz = 0$$

Def: The <u>residue</u> of $f(z)$ at an isolated singularity is the unique complex number R s.t. $f(z) - \frac{R}{z-a}$ is the derivative of an analytic fcn on the punctured disk $\{0 < |z-a| < \delta\}$.

$R = \operatorname{Res}_{z=a} f(z)$.

Let γ be a cycle in Ω' w/ γ homologous to 0 in Ω.

$\gamma \sim \sum_j n(\gamma, a_j) C_j$. wrt Ω'.



$$\int_\gamma f\,dz = \sum_j n(\gamma, a_j) \int_{C_j} f\,dz$$

$$= \sum_j n(\gamma, a_j) P_j$$

Or, $\frac{1}{2\pi i} \int_\gamma f\,dz = \sum_j n(\gamma, a_j) \operatorname{Res}_{z=a_j} f$.

This works for an ∞ # of isolated singularities; the RHS is a finite sum. (b/c limit pt on bd of Ω, so can shrink Ω to contain finitely many)

Thm: Let $f(z)$ be analytic except for isolated singularities $\{a_j\}$ in a region Ω. Then

$$\frac{1}{2\pi i}\int_\gamma f\,dz = \sum_j n(\gamma,a_j)\,\mathrm{Res}_{z=a_j} f(z) \quad \forall \text{ cycles } \gamma$$

which are homologous to 0 in Ω & do not pass through any a_j.

If $n(\gamma,a_j)\in\{0,1\}$, then $\frac{1}{2\pi i}\int_\gamma f\,dz = \sum_{\substack{a_j \leftarrow \text{sings} \\ \text{inside } \gamma}} \mathrm{Res}_{z=a_j} f$

To compute the $\int_\gamma f\,dz$, we need to know how to compute Res.

Suppose $a\in\Omega$ is a pole of f. Then

$$f(z) = B_n(z-a)^{-n} + \cdots + B_1(z-a)^{-1} + \underbrace{\phi(z)}_{\text{analytic in } |z-a|<\delta}$$

Claim: $\mathrm{Res}_{z=a} f = B_1$.

$$f - B_1(z-a)^{-1} = \underbrace{B_n(z-a)^{-n} + \cdots + B_2(z-a)^{-2}}_{\text{derivs of analytic fcns (raise power by 1)}} + \phi(z) \quad \text{Thus antideriv.}$$

on $0<|z-a|<\delta$

So this has an analytic antideriv. on $0<|z-a|<\delta$, & $\int$ cl. curves $=0$, so $B_1 = \mathrm{Res}_{z=a} f(z)$.

If a is a simple pole (ie $f(z) = B_1(z-a)^{-1} + \phi(z)$), then $\mathrm{Res}_{z=a} f = (z-a)f(z)\big|_{z=a}$

Ex: $\frac{e^z}{(z-a)(z-b)} = f(z)$, $a\neq b$. Then

$$\mathrm{Res}_{z=a} f(z) = \lim_{z\to a}(z-a)\frac{e^z}{(z-a)(z-b)} = \frac{e^a}{a-b}$$

Ex: If $a=b$, $f(z) = \frac{e^z}{(z-a)^2}$. ⟵ Taylor series of e^z centered at $z=a$

$$f(z) = \frac{e^a + e^a(z-a) + (z-a)^2\phi(z)}{(z-a)^2}$$

$$= \frac{e^a}{(z-a)^2} + \frac{e^a}{z-a} + \phi(z) \; ; \quad \mathrm{Res}_{z=a} f(z) = e^a$$

# of zeros $= \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz$ $\quad (n(\gamma, a_j) \in \{0,1\})$

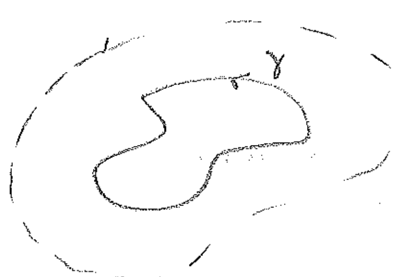
Suppose $f(z)$ has a zero of order h at $z=a$.

Then $f(z) = (z-a)^h f_h(z)$, $f_h(z) \neq 0$ (h any integer)

$f'(z) = h(z-a)^{h-1} f_h(z) + (z-a)^h f_h'(z)$

$$\frac{f'(z)}{f(z)} = \frac{h}{z-a} + \underbrace{\frac{f_h'(z)}{f_h(z)}}_{\neq 0 \text{ at } z=a}$$

So $\operatorname{Res}_{z=a} \frac{f'}{f} = h$.



$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_{\substack{a_j = \text{zeros of } f \\ \text{(counted w/ mult.)}}} n(\gamma, a_j)$$

$$= \sum_{\substack{\text{zeros not} \\ \text{w/ mult.}}} n(\gamma, a_j) \operatorname{Res}_{z=a_j} \frac{f'}{f}$$

(f may have isolated poles)

So, $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \sum_{a_j \text{ zeros of } f} \operatorname{Res}_{z=a_j} \frac{f'}{f} \cdot n(\gamma, a_j) + \sum_{b_j \text{ poles of } f} \operatorname{Res}_{z=b_j} \frac{f'}{f} \cdot n(\gamma, b_j)$

- if f has a zero or a pole at a pt, f'/f has a simple pole at that pt. ($= h$ if zero, $= -h$ if pole)

$$= \sum_{\substack{a_j \text{ zeros of } f \\ \text{w/ mult.}}} n(\gamma, a_j) - \sum_{\substack{b_j \text{ poles of } f \\ \text{w/ mult.}}} n(\gamma, b_j)$$

True for f w/ isolated poles (ie, f is meromorphic), γ a cycle homologous to 0 in Ω, γ not passing through zero or pole of f.

Case to remember: If $n(\gamma, a) \in \{0,1\}$, then

$\frac{1}{2\pi i}\int_\gamma f'/f\,dz$ = # of zeros of f in γ w/ mult − # of poles of f in γ w/ mult.

Recall: Set $\Gamma = f \circ \gamma$, & change of vars:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{dw}{w} = n(\Gamma, 0)$$

(ie, # of zeros − # of poles $= n(\Gamma, 0)$)

<u>Rouche's Thm</u> (Cor of above): Let γ be homo. to 0 in Ω & s.t. $n(\gamma, z) \in \{0,1\}$ $\forall z \notin \gamma$. Suppose $f(z), g(z)$ are analytic in Ω & satisfy

$|f(z) - g(z)| < |f(z)|$ for $z \in \gamma$

Then $f(z)$ & $g(z)$ have the same # of zeros inside of γ.

<u>Pf</u>: f is never zero, on γ else cannot have strict inequality.

$\left|1 - \frac{g(z)}{f(z)}\right| < 1 \Rightarrow g$ never zero, on γ else can't have $<$.

Let $F(z) = \frac{g(z)}{f(z)}$. The values of F on γ lie in $B(1,1)$ (open ball). Let $\Gamma = F \circ \gamma$. We conclude $n(\Gamma, 0) = 0$ (b/c 0 outside of $B(1,1)$).

But $n(\Gamma, 0)$ = # of zeros of F in γ − # of poles of F in γ.

= # of zeros of g in γ − # of zeros of f in γ.

(pole of F = zero of f, & zero of F = zero of g)

$\square$

2/27 Rouché's Thm: Let γ be homologous to 0 in Ω & $n(\gamma, z) \in \{0,1\}$ $\forall z \notin \gamma$. Suppose $|f(z)-g(z)| < |f(z)|$ on γ. Then f & g have the same # of zero's inside γ.

Ex: How many roots does the fcn $g(z) = z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have on the unit disk?

Let $f(z) = 6z^3$. (the biggest term of g). For $|z|=1$

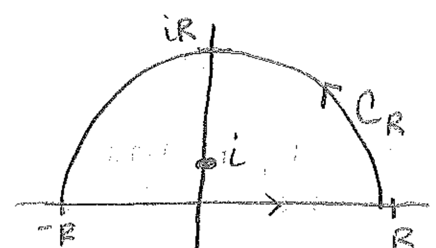
$|f(z)-g(z)| \le |z|^7 + 2|z|^5 + |z| + 1 = 5 < 6|z|^3 = |f(z)|$

& $f(z)$ has 3 zeros at $z=0$, & f, g have same # of zeros in $|z|<1 \Rightarrow g$ has 3 zeros.

$\boxed{\text{Definite Integrals}}$

Ex: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$.

Let $f(z) = \frac{1}{1+z^2}$ $\quad \gamma_R =$



$f(z) = \frac{1}{(z-i)(z+i)}$

$\to$ the only pole inside of γ of $f(z)$ is at $z=i$

$\mathrm{Res}_{z=i} f(z) = \lim_{z \to i} (z-i)\frac{1}{(z-i)(z+i)} = \frac{1}{2i}$

We conclude $\int_{\gamma_R} f(z)dz = \frac{2\pi i}{2i} = \pi$.

Let C_R be the half-circle of radius R.

Claim: $\int_{C_R} f\,dz \to 0$ as $R \to \infty$. Note: for $z \in C_R$, $R \ge 100$,

$|f(z)| = \left|\frac{1}{1+z^2}\right| \le B/|z|^2 = B/R^2$ (for large $|z|$, the 1 doesn't matter)

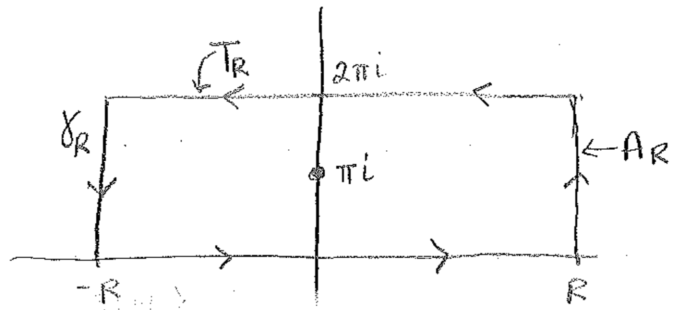
$\left|\int_{C_R} f(z)dz\right| \le \int_{C_R} |f(z)||dz| \le \frac{B}{R^2}\cdot \text{Length of } C_R = \frac{B\cdot\pi R}{R^2} = \frac{B\pi}{R} \xrightarrow{R\to\infty} 0$

$\pi = \lim_{R\to\infty} \int_{\gamma_R} f(z)dz = \lim_{R\to\infty}\left[\int_{-R}^{R} \frac{1}{1+z^2}dz + \int_{C_R} f\,dz\right]$

$= \int_{-\infty}^{\infty} \frac{1}{1+z^2}dz$

Ex (harder): $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin \pi a}, \quad 0<a<1.$

Let $f(z) = \frac{e^{az}}{1+e^{z}}$ & consider



• πi is our only pole in γ_R.

• Note: $(z-\pi i)\, f(z)$
$$= e^{az}\cdot \frac{(z-\pi i)}{1+e^{z}}$$
$$= e^{az}\cdot \frac{z-\pi i}{e^{z}-e^{\pi i}}$$

& $\lim_{z\to\pi i} \frac{e^{z}-e^{\pi i}}{z-\pi i} = e^{\pi i}$ (deriv of e^z at πi)
$= -1$

So, $\lim_{z\to\pi i} (z-\pi i) f(z) = -e^{a\pi i}$ (simple pole, & $-e^{a\pi i}$ its residue)

We conclude $\int_{\gamma_R} f\,dz = -2\pi i\, e^{a\pi i}$

Let $I_R = \int_{-R}^{R} f(x)\,dx$. We want to compute
$I = \lim_{R\to\infty} I_R$.

$\int_{T_R} f(z)dz$ (e^z doesn't see the imaginary part b/c its $2\pi i$ but e^{az} does, but is constant, & in opp dir)
$= -e^{2\pi i a}\, I_R$

Let $A_R = \{R+it \mid 0 \le t \le 2\pi\}$. essentially e^{aR}/e^{R}, the ½ to compensate for the 1+
$$\left|\int_A f\right| \le \int_0^{2\pi} \left|\frac{e^{a(R+it)}}{1+e^{R+it}}\right| dt \le \frac{1}{2}\cdot 2\pi\cdot e^{(a-1)R} \xrightarrow{R\to\infty} 0$$

Same for left side. $(\to 0)$

Taking $R\to\infty$, get
$$-2\pi i\, e^{a\pi i} = I - e^{2\pi i a}\cdot I$$
$$\Rightarrow I = -2\pi i \frac{e^{a\pi i}}{1-e^{2\pi i a}}$$ (let's check this is real:)
$$I = \frac{2\pi i}{e^{\pi i a}-e^{-\pi i a}} = \frac{\pi}{\sin \pi a}$$

Uniform Convergence on Cpt Sets

$f_n : \Omega_n \to \mathbb{C}$

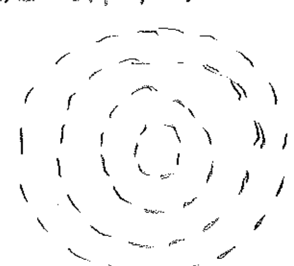
$\lim_{n\to\infty} f_n(z) = f(z) \quad \forall z \in \Omega \quad , \quad f : \Omega \to \mathbb{C}.$

This makes sense for z s.t. $\exists N$ s.t. $n \geq N \Rightarrow z \in \Omega_n$.
- usually will use for Ω_n nested: $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$, & $\Omega = \bigcup \Omega_j$.

$\Omega \subseteq \mathbb{C}$ is open. Want $\forall z \in \Omega \ \exists N$ s.t. $n \geq N$, $z \in \Omega_n$.
If $K \subseteq \Omega$ cpt, $\exists N$ s.t. $n \geq N$ & $z \in K \Rightarrow z \in \Omega_n$.
b/c $\Omega \cap \Omega_n$ is a cover for K, so can pick a finite subcover. (so can ask if conv. unif. on that cpt set)

<u>Thm</u>: Suppose $f_n(z)$ is analytic on Ω_n & f_n converges to a limit fcn $f(z)$ on Ω, uniformly on cpt sets. Then $f(z)$ is analytic on Ω, and $f_n'(z)$ converges to $f'(z)$ unif. on cpt sets.
(continue, & $f_n^{(i)}(z)$ conv. unif on cpt sets to $f^{(i)}(z)$)

Uniform on cpt sets: $\forall K \subseteq \Omega$ cpt $\forall \varepsilon > 0 \ \exists N$ s.t. $\forall z \in K \ \forall n \geq N$, $|f_n(z) - f(z)| < \varepsilon$.
(fix K, choose N independent of $z \in K$ – but does depend on K)



- every cpt set K is eventually in one of the disks b/c the disks become Ω.

Pf #1: Recall Morera's Thm: If $f(z)$ cts on Ω & $\int_\gamma f\,dz = 0$ $\forall$ cl. curves γ, then f is analytic.

Let γ be a closed curve in Ω, So γ is a cpt subset of Ω.

uniform on γ

$$\therefore \int_\gamma f(z)dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz = 0,$$

b/c f_n analytic & so $\int$ around cl. curves is 0.

Redo: Fix $z_0 \in \Omega$. Pick a closed ball $\{|z-z_0| \le r\} \subseteq \Omega$. We will show $f(z)$ is analytic on $\{|z-z_0|<r\}$. Let γ be a cl. curve in $\{|z-z_0|<r\}$. Then

$$\int_\gamma f\,dz = \int_\gamma \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty} \int_\gamma f_n(z)dz = 0.$$

(need to restrict down to a simply ctd set - ie an open ball around z_0).

3/1 p.154/2,3; p.161/1,3a-d,h,i,4 ; p.178/2,4

Suppose we want to show the following is analytic.
$G(z) = \int_{-\infty}^{\infty} F(x,z)\,dx$. For each x, $F(x,\cdot): \Omega \to \mathbb{C}$ & is analytic.
Fix a ball $B(z_0,r) \subseteq \Omega$. We try to show G is analytic on $B(z_0,r)$, since its s.ctd. Fix a closed curve γ in $B(z_0,r)$. Goal: Show $\int_\gamma G(z)dz = 0$.
Idea: $\int_\gamma \int_{-\infty}^{\infty} F(x,z)dx\,dz \overset{\text{Fubini}}{=} \int_{-\infty}^{\infty} \int_\gamma F(x,z)\,dz\,dx = 0$, since F analytic in z. (as long as you can apply Fubini, which you almost always can.

Last time: Suppose $f_n: \Omega_n \to \mathbb{C}$ analytic, $f: \Omega \to \mathbb{C}$, Ω_n & $\Omega \subseteq \mathbb{C}$ open & ctd sets. Suppose $f_n \to f$ uniformly on cpt subsets of Ω. Then f analytic & $f_n' \to f'$ unif. on cpt sets.

Pf #2: Fix $z_0 \in \Omega$. $\{|z_0 - z| \le r\} \subseteq \Omega$. We will show $f_n' \to f'$ unif on $\{|z-z_0| \le r/2\}$ (& we'll be done - b/c cpt set can be covered w/ finitely many balls)
Let C be the circle $|z-z_0| = r$.

$$f_n'(z) = \frac{1}{2\pi i}\int_C \frac{f_n(\zeta)}{(\zeta-z)^2}\, d\zeta \quad \text{(Cauchy's Int. Formula) for } |z-z_0|<r.$$

If $|z-z_0| \le r/2$, the denom. is bdd away from 0 for $\zeta \in C$ (ie, $> r/2$) & $f_n \to f$ unif on C, since C is a cpt subset.
Thus $\frac{1}{2\pi i}\int_C \frac{f_n(\zeta)}{(\zeta-z)^2}\, d\zeta \longrightarrow \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^2}\, d\zeta$ unif for $|z-z_0| \le r/2$. □

Cor: If a series $f(z) = f_1(z) + f_2(z) + \cdots$ ($f_i(z)$ analytic) conv. unif. on every cpt subset of a region Ω, then $f(z)$ is analytic on Ω, & the series can be differentiated term by term.

Thm (Hurwitz's Thm): If the fcns $f_n(z)$ are analytic & never 0 in a region Ω & $f_n \to f$ unif. on cpt subsets of Ω, then either $f \equiv 0$ or f is never 0 on Ω.

Pf: Suppose $f \not\equiv 0$. The zeros of f are isolated. Fix $z_0 \in \Omega$. f is non-zero on the punctured disk $0 < |z-z_0| \le r$ (either by continuity or isolation of zeros). In particular, $|f(z)|$ has a positive min. on $|z-z_0| = r$ (call this circle C).
$\frac{1}{f_n(z)} \to \frac{1}{f(z)}$ unif. on C. We also know $f_n'(z) \to f'(z)$ unif. on C. Thus $\lim_{n\to\infty} \frac{1}{2\pi i}\int_C \frac{f_n'(z)}{f_n(z)}\, dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz$
LHS = # of zeros of f_n in $C = 0$ = # of zeros of f in C = RHS □

Taylor Series

$|z-z_0| \le \rho \subseteq \Omega$

(ie, closure is in Ω)

On the disk,

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + f_{n+1}(z)(z-z_0)^{n+1}$$

$$f_{n+1}(z) = \frac{1}{2\pi i}\int_{C=\{|z-z_0|=\rho\}} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}(\zeta-z)}\,d\zeta$$

The error term is actually pretty small.

Let $M = \max_{z \in C} |f(z)|$ (exists b/c cts fcn on cpt set)

$$\left| f_{n+1}(z)(z-z_0)^{n+1} \right| \le \frac{1}{2\pi} \frac{M \cdot |z-z_0|^{n+1}}{\rho^{n+1}(\rho - |z-z_0|)} \cdot 2\pi\rho$$

$$= \frac{M|z-z_0|^{n+1}}{\rho^n(\rho - |z-z_0|)}$$

Suppose $|z-z_0| \le r < \rho$

$\to 0$ uniformly for $|z-z_0| \le r$ $\left(\frac{r^{n+1}}{\rho^n \ldots} \to 0\right)$

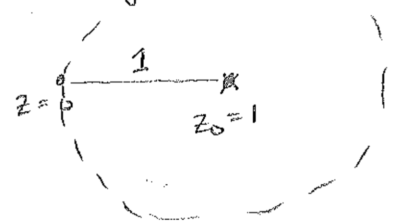
As $n \to \infty$, the error term of the taylor series conv. unif. on cpt. subsets of $|z-z_0| < \rho$. ρ could be anything $< \operatorname{dist}(z_0, \partial\Omega)$. So, as $n \to \infty$, the error term of the Taylor series $\to 0$ unif. on cpt. subsets of $|z-z_0| < \operatorname{dist}(z_0, \partial\Omega)$

(ie, error term $\to 0$ in largest disk can draw around z_0 in Ω)

Thm: If $f(z)$ is analytic in a region Ω containing z_0, then $f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots$ is valid for z in the largest open disk centered at z_0 inside Ω.

(ie, the power series conv. on this disk — radius of conv is at least as large as the dist. to $\partial\Omega$)

Ex: $\frac{1}{z}$ analytic except at $z=0$. So radius of conv. of T. series at $z_0 = 1$ conv. for rad. 1.



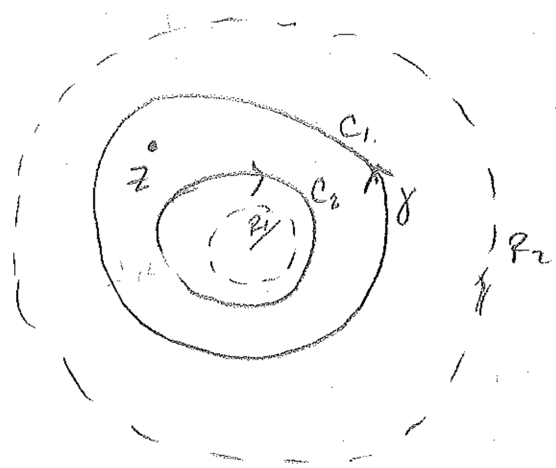
Series of the form
$b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots$, can be thought of as a power series in $w = 1/z$. Conv. in w for a disk, so conv in z outside a circle $|z| = R$ (could be $R=\infty$). Will conv. unif. in every region $|z| \geq \rho$ where $\rho > R$ (b/c $1/$that will be conv. for cpt. sets in w).

Add a power series in z & get

$\sum_{n=-\infty}^{\infty} a_n z^n$. This is a sum of non-neg. powers & neg. powers. The non-neg. powers conv. for $|z| < R_2$, The neg. powers conv. for $|z| > R_1$.
If $R_1 < R_2$, this defines an analytic fcn on the annulus $R_1 < |z| < R_2$. ($R_1 = 0$ & $R_2 = \infty$ are possibilities).
In fact, converse is true! Start w/ an analytic fcn $f(z)$ on $R_1 < |z-a| < R_2$. We show $f(z) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n$ converges unif. on cpt subsets of $R_1 < |z-a| < R_2$
Called the <u>Laurent Series</u>

*remember proof!

Pf: We want $f(z) = f_1(z) + f_2(z)$ where $f_1(z)$ analytic for $|z-a| < R_2$ (b/c then has a power series that conv. unif. on cpt subsets) & $f_2(z)$ is analytic for $|z-a| > R_1$ w/ a removable singularity at ∞ (ie $f_2(1/z)$ has remov. sing at 0) (will get power series in neg. powers of $z-a$).



Want to define f_1 & f_2 at z.

- Pick 2 circles as shown

$\gamma = C_1 + C_2$

Claim: $\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z)$

(Cauchy's Thm)

γ can be one closed curve (w/ seg. that cancels out)

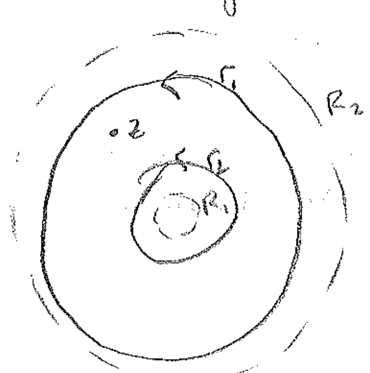
$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

3/4 Let $f(z)$ be analytic on $R_1 < |z-a| < R_2$.

$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n$, w/ uniform conv. on cpt. sets of the annulus.

Called the Laurent series.

We will show $f(z) = f_1(z) + f_2(z)$, f_1 analytic in $|z-a| < R_2$, & f_2 analytic in $|z-a| > R_1$, f_2 w/ removable sing at ∞.



$$f_1(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=r_1} \frac{f(\zeta)\, d\zeta}{\zeta - z}, \quad |z-a| < r_1 < R_2$$

$$f_2(z) = -\frac{1}{2\pi i}\int_{|\zeta-a|=r_2} \frac{f(\zeta)\, d\zeta}{\zeta - z}, \quad R_1 < r_2 < |z-a|$$

$f(z) = f_1(z) + f_2(z)$ by Cauchy's thm

The def. is ind. of r_1, r_2 so long as the ineq's are satisfied.

$f_1(z)$ is analytic on $|z-a| < R_2$ as desired, & $f_2(z)$ analytic in z on $|z-a| > R_1$. For z large,

$$|f_2(z)| \le \frac{1}{2\pi}\left(\max_{|\zeta-a|=r_2} |f(\zeta)|\right) \cdot \frac{1}{2}\frac{1}{|z|} \cdot 2\pi r_2 \to 0 \text{ as } |z| \to \infty$$

$f(z)=\sum_{n=-\infty}^{\infty} A_n(z-a)^n$, where $f_1(z)=\sum_{n=0}^{\infty} A_n(z-a)^n$ &

$f_2(z)=\sum_{n=-\infty}^{-1} A_n(z-a)^n$

Let's compute A_n:

For f_1: $A_n = \frac{f_1^{(n)}(a)}{n!} = \frac{1}{n!}\cdot\frac{1}{2\pi i}\int_{|\zeta-a|=r}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta$

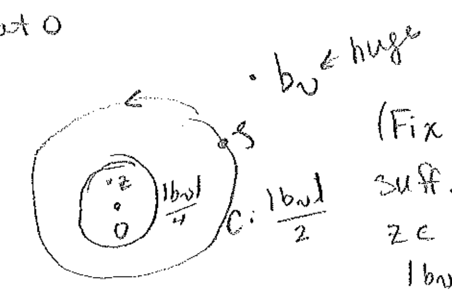
For f_2: $z \longmapsto a+\frac{1}{z'}$, same computation. Get same formula for A_n.

<u>Thm</u> (Mittag-Lefler Thm): Let $\{b_\nu\}$ be a seq. of $\mathbb{C}$ numbers w/ $\lim_{\nu\to\infty} b_\nu=\infty$. Let $P_\nu(\zeta)$ be polys w/o constant terms. Then there are fcns which are meromorphic on $\mathbb{C}$ with poles at b_ν & the singular part $P_\nu\left(\frac{1}{z-b_\nu}\right)$. Any mero fcn of this type can be written as $f(z)=\sum_\nu\left[P_\nu\left(\frac{1}{z-b_\nu}\right)-p_\nu(z)\right]+g(z)$, where $p_\nu(t)$ are polys & g is entire.
(p_ν's necessary to ensure sums converge)

We may suppose no b_ν is 0. The fcn $P_\nu\left(\frac{1}{z-b_\nu}\right)$ is analytic at $z=0$ & for $|z|<b_\nu$. We expand it in a Taylor series about zero: We let p_ν be a partial sum of the Taylor series of $P_\nu\left(\frac{1}{z-b_\nu}\right)$ up to order n_ν.

$f(z)=f(0)+f'(0)z+\cdots+\frac{f^{(n-1)}(0)}{(n-1)!}z^{n-1}+z^n f_n(z)$. ← holo.

$f_n(z)=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{\zeta^n(\zeta-z)}\,d\zeta$.
↖ circle centered at 0



(Fix z, choose ν suff. large s.t. $z\in$ circle w/ rad. $\frac{|b_\nu|}{4}$)

$M_\nu = \max_{|z|\le \frac{|b_\nu|}{2}} \left| P_\nu\left(\frac{1}{z-b_\nu}\right)\right|$ (const. depending on ν - ind. of n_ν)

$$\left|P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right| \le |z|^{n_\nu+1} \frac{1}{2\pi} M_\nu \cdot \frac{2\pi\, |b_\nu|/2}{\left(\frac{|b_\nu|}{2}\right)^{n_\nu+1}\left(\frac{|b_\nu|}{2}\right)} \quad \forall\, |z| < \frac{|b_\nu|}{4}$$

$p_\nu(z)$: 1st n_ν terms of Taylor series : $n-1 = n_\nu$

$$= C_\nu \left(\frac{|z|}{(|b_\nu|/2)}\right)^{n_\nu+1} \le \frac{1}{2^\nu}$$ by taking n_ν large, which conv.' in ν.

C_ν: const

Given a cpt. set K, pick ν so large that for $\nu' \ge \nu$,
$K \subseteq \{|z| < |b_\nu|/4\}$

$\sum_{\nu' \ge \nu} \left[P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right]$ conv. unif. on K (b/c on $K \le \frac{1}{2^\nu}$)
$\to$ analytic on $|z| < |b_\nu|/4$.

$\sum_{\nu' < \nu} \left[P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right]$ is mero. on $\mathbb{C}$ w/ poles at b_ν
w/ sing. part $P_\nu\left(\frac{1}{z-b_\nu}\right)$.

We conclude $\sum_\nu \left[P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right]$ is mero, as desired.

$\square$

Now, want an entire fcn with proscribed zeros.
Given $\{b_\nu\}$, want them to be zeros of an entire fcn.
Finite case: b_1, b_2, b_3 : $(z-b_1)(z-b_2)(z-b_3)e^{g(z)}$, $g(z)$ entire
(b/c can mult by fcn never zero $\Rightarrow$ is exp. of an entire fcn).
If $F(z)$ had precisely the zeros b_1, b_2, b_3, then
$\frac{1}{(z-b_1)(z-b_2)(z-b_3)} \cdot F(z)$ would never be zero, so it would $= e^{g(z)}$.

ie, these are the only fcns w/ zeros b_1, b_2, b_3, so if
know one fcn w/ zeros, know all fcns w/ zeros
b/c they only differ by a $g(z)$.

Infinite Products of $\mathbb{C}$ #'s

$p_1 p_2 p_3 \cdots$, $p_j \in \mathbb{C}$,

$= \prod_{j=1}^{\infty} p_j$

$\prod_{j=1}^{\infty} p_j = \lim_{N\to\infty} \prod_{j=1}^{N} p_j$. Not enough. $p_j = 1/2$. $\prod_{j=1}^{N} 1/2 = 1/2^N \to 0$

Don't want this.

$\prod_{j=1}^{\infty} p_j$ is said to converge if only finitely many of the p_j's are 0, & if the p_j' are p_j's w/o zero terms, $\lim_{N\to\infty} \prod_{j=1}^{N} p_j'$ exists & is non-zero.

- If there are zero terms, prod = 0,
- If no zero terms, prod = Limit.

Let $P_N = \prod^{N} p_j$ (no zeros)

$\frac{P_N}{P_{N-1}} = p_N \xrightarrow{N\to\infty} 1$ if product converges (necessary)

3/6 Let $\{p_n\}$ be a sequence of non-zero $\mathbb{C}$ #'s. We say $\prod_{n=1}^{\infty} p_n$ converges if $\lim_{N\to\infty} \prod_{n=1}^{N} p_n$ exists & is non-zero. If a finite # of the p_n's are zero, we say $\prod_{n=1}^{\infty} p_n$ conv. if the prod. of the non-zero terms.

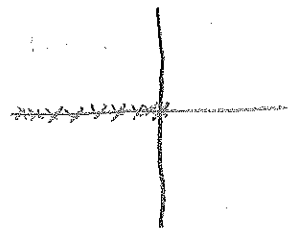
$\prod_{n=1}^{\infty} p_n$ conv $\Rightarrow$ $p_n \to 1$

$p_n = 1 + a_n$, $a_n \to 0$. log will turn the prod into a sum.

Recall:

$\log z = \log|z| + i\arg z$

angle from real axis, well-def. up to mult. of 2π



$\mathrm{Log}\, z = \log|z| + i\,\mathrm{Arg}\, z$
principle Branch
$-\pi < \mathrm{Arg}\, z \le \pi$
(agrees w/ real log on positive real axis)

cts. everywhere except neg. real axis — cut out.

Look at $\prod_1^\infty (1+a_n)$. compare to $\sum_1^\infty \mathrm{Log}(1+a_n)$
We stay in the case where no $P_n = 0$.
for large n, a_n close to 0 so far from discont.

$P_N = \prod_1^N (1+a_n)$, $S_N = \sum_1^N \mathrm{Log}(1+a_n)$

$$e^{S_N} = P_N$$

So if $S_n \to S$, then $P_N \to e^S \neq 0$, ie, $\prod_1^\infty (1+a_n) = e^S$

The converse is true: Call $P = \lim P_N$

$\sum_{n=1}^\infty \mathrm{Log}(1+a_n)$ is not necessarily $\mathrm{Log}\, P$. It may be some other $\log P$.

Since $\frac{P_N}{P} \to 1$, $\mathrm{Log}(P_N/P) \to 0$.

$\exists\, h_n \in \mathbb{Z}$ s.t. $\mathrm{Log}(P_N/P) = S_n - \mathrm{Log}\, P + h_n \cdot 2\pi i$

$\Rightarrow (h_{n+1} - h_n) 2\pi i = \underset{\downarrow\atop 0}{\mathrm{Log}\left(\frac{P_{n+1}}{P}\right)} - \underset{\downarrow\atop 0}{\mathrm{Log}\left(\frac{P_n}{P}\right)} - \underset{\downarrow\atop \text{Im part btwn } -\pi \,\&\, \pi}{\mathrm{Log}(1+a_n)}$

We conclude $h_n = h_{n+1}$ for n large, ie, h_n is eventually constant $h \in \mathbb{Z}$.

$\mathrm{Log}(P_n/P) = S_n - \mathrm{Log}\, P + h \cdot 2\pi i$

as $n \to \infty$, we get $\lim_{n\to\infty} S_n = \mathrm{Log}\, P - h 2\pi i$ $\square$

<u>Thm</u>: For $p_n \neq 0$, $\prod_{n=1}^{\infty}(1+a_n)$ converges iff $\sum_{n=1}^{\infty} \mathrm{Log}(1+a_n)$ converges.

$*$ If a_n depended on z, then $\prod_{n=1}^{\infty}(1+a_n(z))$ conv. uniformly in z iff $\sum_{n=1}^{\infty} \mathrm{Log}(1+a_n(z))$ conv. unif in z.

We say $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely if $\sum_{n=1}^{\infty} \mathrm{Log}(1+a_n)$ conv. absolutely.

<u>Note</u>: $\lim_{z\to 0} \underbrace{\frac{\mathrm{Log}(1+z)}{z}}_{\text{diff. quotient at } 1} = 1$

So $\underbrace{\sum_{n=1}^{\infty} |\mathrm{Log}(1+a_n)|}_{\text{approx } |a_n| \text{ from limit} = 1 \text{ above.}} < \infty \iff \sum_{n=1}^{\infty} |a_n| < \infty$ (only true w/ abs. val.)

If $g(z)$ is entire, then $f(z) = e^{g(z)}$ is entire & never 0.
Conversely, if $f(z)$ is entire & never 0 then $f(z) = e^{g(z)}$ for some $g(z)$ entire.

Assume that $f(z)$ has m zeros at zeros & zeros $a_1, \dots, a_N$ all non-zero, listed with multiplicity.
Then $f(z) = z^m e^{g(z)} \prod_1^N (1 - z/a_n)$
If we had ∞-ly many zeros, we could try $f(z) = z^m e^{g(z)} \prod_1^{\infty} (1 - z/a_n)$. This would work if prod. conv. unif. on cpt. sets. (This is true iff $\sum \frac{1}{|a_n|} < \infty$)

Let $\{a_n\}^{\neq 0}$ be any seq. of $\mathbb{C}$ numbers w/ $\lim_{n\to\infty} a_n = \infty$.
We will make a fcn w/ these zeros.
We want polynomials $P_n(z)$ s.t.
$\prod_1^{\infty} \left[\left(1 - \frac{z}{a_n}\right) e^{P_n(z)} \right]$ conv. to an entire fcn.

(the $e^{p_n(z)}$ is a factor to help things conv.)

We want to consider

$r_n(z) = \text{Log}(1 - \frac{z}{a_n}) + p_n(z)$

Goal: show $\sum r_n(z)$ conv. unif. on cpt sets ($\Rightarrow$ prod. conv.)

Fix R. WTS $\sum |r_n(z)|$ conv. unif. for $|z| \le R$.

Consider only terms w/ $|a_n| > R$ (drops off finite # of terms $\to$ doesn't change conv).

If $|z| \le R$, $\text{Log}(1 - \underset{\substack{\uparrow \\ <1}}{\frac{z}{a_n}}) = \frac{-z}{a_n} - \frac{1}{2}(\frac{z}{a_n})^2 - \frac{1}{3}(\frac{z}{a_n})^3 - \cdots$

} taylor series of $\text{Log}(1-w)$ conv. for $|w|<1$

Choose $p_n(z) = \frac{z}{a_n} + \frac{1}{2}(\frac{z}{a_n})^2 + \cdots + \frac{1}{m_n}(\frac{z}{a_n})^{m_n}$

So $r_n(z) = -\frac{1}{m_n+1}(\frac{z}{a_n})^{m_n+1} - \frac{1}{m_n+2}(\frac{z}{a_n})^{m_n+2} - \cdots$

$$|r_n(z)| \le \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1}\left(1 - \frac{R}{a_n}\right)^{-1}$$

b/c $|r_n(z)| \le \frac{1}{m_n+1}\left(\frac{|z|}{|a_n|}\right)^{m_n+1}\left(1 + \frac{|z|}{|a_n|} + \left(\frac{|z|}{|a_n|}\right)^2 + \cdots\right)$, $|z| \le R$

If $\sum_n \frac{1}{m_n+1}\left(\frac{R}{|a_n|}\right)^{m_n+1}$ conv, then $\sum |r_n(z)|$ conv. unif. for $|z| \le R$.

Choose $m_n = n$. Then $\sum_n \frac{1}{n+1}\left(\frac{R}{|a_n|}\right)^{n+1}$ ($|a_n| > R$) is a geometric series since $\frac{R}{|a_n|} < c < 1$, so it converges.

We conclude $\sum r_n(z)$ conv. unif. for $|z| \le R$.

<u>Thm</u>: $\exists$ an entire fcn w/ arbitrary proscribed zeros, a_n, so long as $a_n \to \infty$. Any such fcn w/ exactly these zeros is of the form

$$f(z) = z^m e^{g(z)} \prod_{\substack{n=1 \\ a_n \ne 0}}^{\infty} (1 - z/a_n) e^{\frac{z}{a_n} + \frac{1}{2}(\frac{z}{a_n})^2 + \cdots + \frac{1}{m_n}(\frac{z}{a_n})^{m_n}}$$ for some $m_n \in \mathbb{N}$

3/8 Thm: There exists an entire fcn w/ arbitrarily proscribed zeros $\{a_n\}$ so long as $\lim_{n\to\infty} a_n = \infty$. Every entire fcn w/ these & no other zeros can be written as

$$f(z) = z^m e^{g(z)} \prod_{\substack{n=1 \\ a_n \neq 0}}^{\infty} \left(1-\frac{z}{a_n}\right) e^{z/a_n + \frac{1}{2}(z/a_n)^2 + \cdots + \frac{1}{m_n}(z/a_n)^{m_n}}$$

where $g(z)$ entire & the m_n are certain integers.

(if $\sum \frac{1}{a_n} < \infty$, don't need $e^{\cdots}$ at all; the prod. converges)

Cor: Every fcn $F(z)$ which is mero. in $\mathbb{C}$ is the quotient of 2 entire fcns $f(z)/g(z)$.

Pf: Let $g(z)$ be an entire fcn whose zeros are the poles of $F(z)$. Then $g(z)\cdot F(z) = f(z)$ is entire. & $F(z) = f(z)/g(z)$ as desired.

It would be nice if we could take all the m_n's equal, say $m_n = h$. This works precisely when $\sum \frac{1}{|a_n|^{h+1}} < \infty$, by the proof we did before. So, can choose least h s.t. sum above is finite.

If h is the smallest int. s.t. $\sum \frac{1}{|a_n|^{h+1}} < \infty$, we call

$$z^m \prod_{\substack{n=1 \\ a_n \neq 0}}^{\infty} \left(1-\frac{z}{a_n}\right) e^{z/a_n + \frac{1}{2}(z/a_n)^2 + \cdots + \frac{1}{h}(z/a_n)^h}$$

the <u>canonical product</u> associated to $\{a_n\}$.

Riemann Mapping Thm

<u>Topology on Spaces of fcns:</u>

$f_n \to f$, uniform conv. on cpt sets preserves continuity & analyticity. Want to define top on fcns s.t. conv. in top is equiv. to unif. conv. on cpt. sets.

Fix $\Omega \subseteq \mathbb{C}$ open. Consider $f: \Omega \to S$, S a metric space w/ metric d.

Recall, $f: \Omega \to S$ is cts at $z_0 \in \Omega$ if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow d(f(z), f(z_0)) < \varepsilon$.

Cts fcns on cpt sets are uniformly cts.

Let $\mathcal{F}$ be a family of fcns $\Omega \to S$.

<u>Def</u>: The fcns in $\mathcal{F}$ are said to be <u>equicontinuous</u> on a cpt set $K \subseteq \Omega$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \underbrace{\forall z, z_0 \in K}_{\text{unif. on } K}, \underbrace{\forall f \in \mathcal{F}}_{\text{unif. on } \mathcal{F}}, |z - z_0| < \delta \Rightarrow d(f(z), f(z_0)) < \varepsilon.$$

<u>Def</u>: A family $\mathcal{F}$ is said to be <u>normal</u> if every sequence $\{f_n\}$ of fcns in $\mathcal{F}$ contains a subseq. which converges unif. on cpt subsets of Ω.

Want: $f_n \to f$ in top. $\Leftrightarrow$ $f_n \to f$ unif. on cpt sets.

cpt in metric space means every <u>sequence</u> has a conv. subseq.

We'll see being normal $\Leftrightarrow$ $\overline{\mathcal{F}}$ is cpt. (at least if everything complete)

A metric topology on cts fcns $\Omega \to S$:

Fix an exhaustion of Ω of cpt sets, ie $E_1 \subseteq E_2 \subseteq \cdots \subseteq \Omega$ w/ E_j cpt & $\bigcup E_j = \Omega$. (def. will be same for any one you pick)

For $f, g : \Omega \to S$ cts, define a metric $\rho(f,g)$. Give S the metric $\delta(a,b) = \frac{d(a,b)}{1 + d(a,b)}$. This induces the same topology on S as d. $\delta(a,b) < 1 \ \forall a, b$.

Set $\delta_k(f,g) = \sup_{z \in E_k} \delta(f(z), g(z))$. Conv. in δ_k is exactly unif. conv. on E_k.

Define $\rho(f,g) = \sum 2^{-k} \delta_k(f,g)$. since 2^{-k} summable.

If $\delta_k(f,g) \to 0 \ \forall k$, by DCT, $\rho(f,g) \to 0$.

Conversely, if $\rho(f,g) \to 0$, $\delta_k(f,g) \to 0 \ \forall k$ (b/c $\rho(f,g) \geq 2^{-k} \delta_k(f,g) \ \forall k$)

• If S is complete, then so is ρ.
Topology is called the <u>topology of unif. conv. on cpt sets</u>.

$\mathcal{F}$ is normal: every seq. $\{f_n\}$ in $\mathcal{F}$ has a subseq. which conv. unif. on cpt sets.
i.e., $\mathcal{F}$ is normal iff $\overline{\mathcal{F}}$ is cpt. in this topology.
(ie, $\mathcal{F}$ is <u>precpt</u> or <u>relatively cpt</u>)

Since $\overline{\mathcal{F}}$ is cpt, $\mathcal{F}$ is <u>totally bdd</u>: $\forall \varepsilon > 0$ $\exists$ finite set $f_1, \dots, f_n \in \mathcal{F}$ s.t. $\forall f \in \mathcal{F}$, $\rho(f, f_j) < \varepsilon$ for some f_j.
(can cover closure w/ balls of radius ε around f's, take finite subcover $\Rightarrow$ your f_j's).
Conversely, if ρ is complete, close and totally bdd $\Rightarrow$ cpt.
If S is complete, then $\mathcal{F}$ is normal iff $\mathcal{F}$ is totally bdd.

<u>Thm</u>: The family $\mathcal{F}$ is totally bdd iff to every cpt set $E \subseteq \Omega$ & all $\varepsilon > 0$ it is possible to find $f_1, \dots, f_n \in \mathcal{F}$ s.t. every $f \in \mathcal{F}$ satisfies
$d(f(z), f_j(z)) < \varepsilon$ $\forall z \in E$ & some f_j.
(when you consider small distances, d & δ are almost the same)

<u>Pf</u>: ($\Rightarrow$) If $\mathcal{F}$ is totally bdd $\exists f_1, \dots, f_n$ s.t. $\forall f \in \mathcal{F}$, $\rho(f, f_j) < \varepsilon$ for some f_j. Fix $E \subset \Omega$ cpt. $E \subseteq E_k$.
We have $\delta_k(f, f_j) < 2^k \varepsilon$ (recall $\rho(f, f_j) = \sum 2^{-k} \delta_k(f, f_j)$)
We can make $\delta_k(f, f_j)$ as small as we wish.
$\delta_k(f, g) = \sup_{z \in E_k} \delta(f(z), g(z))$, $\delta(a,b) = \frac{d(a,b)}{1 + d(a,b)}$
δ_k small $\Rightarrow$ δ small $\Rightarrow$ d small
So $\sup_{z \in E_k} d(f(z), f_j(z))$ is small for some j.

3/11 Let $\Omega \subseteq \mathbb{C}$ open & (S,d) a metric sp. Let $\mathcal{F}$ be a family of cts fcns $\Omega \to S$. $\mathcal{F}$ is said to be an equicontinuous family on $K \subseteq \Omega$ cpt if $\forall \varepsilon > 0\ \exists \delta > 0$ $\forall z, z_0 \in K$, $\forall f \in \mathcal{F}$, $|z - z_0| < \delta \Rightarrow d(f(z), f(z_0)) < \varepsilon$.

- δ is ind. of z_0 (unif. cts) & ind. of f.

$\mathcal{F}$ is said to be normal in Ω if every sequence $\{f_n\} \subseteq \mathcal{F}$ contains a subseq f_{n_k} which conv. unif. on cpt sets.

We saw last time: $\mathcal{F}$ is normal iff $\mathcal{F}$ is relatively cpt in the top. of unif. conv. on cpt. sets.

In this case, $\mathcal{F}$ is totally bdd,
(totally bdd is equiv. to rel. cpt. if S is complete)

Recall: We defined a new metric on S, $\delta(x,y) = \frac{d(x,y)}{1 + d(x,y)}$ (bdd by 1)

Took $E_1 \subset E_2 \subset \cdots \subset \Omega$ cpt, $\Omega = \cup E_j$, defined
$\delta_k(f,g) = \sup_{z \in E_k} \delta(f(z), g(z))$ (conv. in δ_k is unif conv. in E_k)

$\rho(f,g) = \sum_k 2^{-k} \delta_k(f,g)$. $\rho(f_n, f) \to 0$ iff $f_n \to f$ unif. on cpt sets.

totally bdd: $\forall \varepsilon > 0\ \exists f_1, \ldots, f_n \in \mathcal{F}$ s.t. $\forall f \in \mathcal{F}$, $\rho(f, f_j) < \varepsilon$ for some f_j.

<u>Thm</u>: $\mathcal{F}$ is totally bdd iff $\forall$ cpt set $E \subseteq \Omega$ & every $\varepsilon > 0$, $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t. $\sup_{z \in E} d(f(z), f_j(z)) < \varepsilon$ for some f_j, $\forall f \in \mathcal{F}$.

($\Rightarrow$)
<u>Pf</u>: If $\mathcal{F}$ is tot. bdd, $\exists f_1, \ldots, f_n \in \mathcal{F}$ s.t. $\forall f \in \mathcal{F}$, $\rho(f, f_j) < \varepsilon$ for some f_j. $\Rightarrow \delta_k(f, f_j) < \varepsilon 2^k$ $\forall k$.
Take k so large that $E_k \supseteq E$. Then $\delta(f(z), f_j(z)) < \varepsilon 2^k$ $\forall z \in E$. So this fixed k, we can make $d(f(z), f_j(z))$ as small as we like.

($\Leftarrow$): Fix $\varepsilon>0$, take k_0 s.t. $2^{-k_0}<\varepsilon/2$. We can find $f_1,\dots,f_n$ s.t. $\forall f\in\mathcal{F}$, $\exists j$ s.t. $\delta(f,f_j)\le d(f,f_j)<\varepsilon/2k_0$ on E_{k_0}.
$\delta_k(f,f_j)<\varepsilon/2k_0$ $\forall k\le k_0$. $\delta_k(f,f_j)<1$ $\forall k$. Thus
$\rho(f,f_j)=\sum 2^{-k}\delta_k(f,f_j)<k_0\,\varepsilon/2k_0+\underbrace{2^{-k_0-1}+2^{-k_0-2}+\cdots}_{\text{bdd by }1}$

$=\varepsilon/2+2^{-k_0}<\varepsilon/2+\varepsilon/2=\varepsilon$ $\square$

Thm (Arzela-Ascoli): A family of cts fcn $\mathcal{F}$ w/ values in a metric sp. S is normal in Ω iff
(i) $\mathcal{F}$ is equicts on every cpt subset $E\subset\Omega$
(ii) $\forall z\in\Omega$, the set $E_z=\{f(z)\mid f\in\mathcal{F}\}$ is precpt in S.

Pf: ($\Rightarrow$) Assume $\mathcal{F}$ is normal. Fix $E\subset\Omega$ & $\varepsilon>0$.
Let $f_1,\dots,f_n\in\mathcal{F}$ be such that $\forall f\in\mathcal{F}$, $d(f,f_j)<\varepsilon$ on E for some j. $f_1,\dots,f_n$ are unif. cts on E. $\exists\delta>0$ s.t. $\forall z,z_0\in E$; $|z-z_0|<\delta\Rightarrow d(f_j(z),f_j(z_0))<\varepsilon$ $\forall j=1,\dots,n$. (since finite, can pick one δ) $\forall f\in\mathcal{F}$, Let f_j be such that $d(f,f_j)<\varepsilon$ on E.
Let $z,z_0\in E$ w/ $|z-z_0|<\delta$.

$$d(f(z),f(z_0))\le d(f(z),f_j(z))+d(f_j(z),f_j(z_0))+d(f_j(z_0),f(z_0))$$
$$<\ \varepsilon+\varepsilon+\varepsilon=3\varepsilon.$$

$\Rightarrow$ (i) equicts.
Let $E_z=\{f(z)\mid f\in\mathcal{F}\}$. Suppose $\{w_n\}\subseteq E_z$. (want a convergent subseq.) Let $f_n\in\mathcal{F}$ s.t. $w_n=f_n(z)$. By normality, $\exists$ a subseq, f_{n_j}, of f_n's that conv. univ. on cpt set. So $w_n=f_n(z)$ converges &
$\therefore \overline{E_z}$ is cpt. $\Rightarrow$ (ii) precpt.
($\Leftarrow$): Let $\{\zeta_k\}\subseteq\Omega$ be a countable dense set.
Let $\{f_n\}\subseteq\mathcal{F}$. (want a conv. subseq).
From $\{f_n\}$, we can extract a conv. subseq. $f_n(\zeta_k)$ for each k. We can pick sequences
$n_{11}<n_{12}<\cdots<n_{1j}<\cdots$
$n_{21}<n_{22}<\cdots<n_{2j}<\cdots$
$\vdots\qquad\qquad\vdots$

For the 1^{st} sequence, pick n_{1j} s.t. $f_{n_{1j}}(\zeta_1)$ conv.
Recursively, pick n^{th} seq. to be a subseq. of the $n-1^{st}$ seq. s.t. $f_{n_{nj}}(\zeta_n)$ conv. Set $n_j = n_{jj}$
Take the diagonal sequence $f_{n_j}(\zeta_j)$. This is eventually a subseq of all sequences, so it conv. $\forall j$.
Let $E \subset \Omega$. Suppose $\mathcal{F}$ equicts. on E. We show f_{n_j} unif. on E. Given $\varepsilon > 0$, Choose $\delta > 0$ s.t. $\forall$ $z, z' \in E$, $f \in \mathcal{F}$, $|z - z'| < \delta \Rightarrow d(f(z), f(z')) < \varepsilon/3$.
Because E cpt, it can be covered by a finite # of $\delta/2$ balls. Select a ζ_k in each ball (can do so b/c ζ_k dense). $\exists$ i_0 s.t. $i, j > i_0$ we have $d(f_{n_i}(\zeta_k), f_{n_j}(\zeta_k)) < \varepsilon/3$ (f_{n_j} conv $\Rightarrow$ Cauchy, i_0 ind. of ζ_k b/c finitely many ζ_k).
For each $z \in E$, one of the ζ_k is within dist. δ of z. Use equicontinuity:
$d(f_{n_i}(z), f_{n_i}(\zeta_k)) < \varepsilon/3$ & $d(f_{n_j}(z), f_{n_j}(\zeta_k)) < \varepsilon/3$

$$d(f_{n_i}(z), f_{n_j}(z)) \le d(f_{n_i}(z), f_{n_i}(\zeta_k)) + d(f_{n_i}(\zeta_k), f_{n_j}(\zeta_k)) + d(f_{n_j}(z), f_{n_j}(\zeta_k))$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

So $f_{n_j}(z)$ is Cauchy unif. for $z \in E$. We conclude f_{n_j} conv. unif on E (b/c cpt) as desired. $\square$

*For a family $\mathcal{F}$ of cts fcns $\Omega \xrightarrow{\subseteq \mathbb{C}} \mathbb{C}$, $\mathcal{F}$ is normal iff
(i) $\mathcal{F}$ equicts on cpt sets
(ii) $\forall z \in \Omega$, $\{f(z) \mid f \in \mathcal{F}\}$ is bdd

* Suppose $\mathcal{F}$ a family satisfying the above 2.
For $z_0 \in \Omega$, let $\rho > 0$ be s.t. $\{|z - z_0| \le \rho\} \subseteq \Omega$.
$\mathcal{F}$ is equicts on this disk. Fix $\varepsilon > 0$, let $\delta > 0$ be as in def. of equicont. Then if $|f(z_0)| < M$, $\forall f \in \mathcal{F}$, then $\forall$ $|z - z_0| < \delta$, $|f(z)| < M + \varepsilon$. We conclude, $\forall z_0 \in \Omega$ $\exists$ disk $|z - z_0| < \delta$ on which $\mathcal{F}$ is uniformly bdd. But every cpt set is covered by a finite # of these disks,

so $\mathcal{F}$ is bdd on cpt sets (ie, Locally bdd)

Thm: A family of analytic fcns is normal iff the fcns in $\mathcal{F}$ are unif. bdd. on every cpt set.
(ie, if analytic, equicts is gotten thru writing deriv. as $\int$ of fcn)

3/13 $\mathcal{F}$ a family of cts fcns $\Omega \subseteq \mathbb{C}$ open $\to \mathbb{C}$.

- We say $\mathcal{F}$ is locally bdd if $\forall z \in \Omega$, $\exists$ nbhd $U \ni z$ s.t. $\sup\{|f(w)| \mid w \in U, f \in \mathcal{F}\} < \infty$.
  $\Leftrightarrow$ $\forall E \subseteq \Omega$ cpt, $\sup\{|f(w)| \mid w \in E, f \in \mathcal{F}\} < \infty$.

Thm: (S, d) a metric sp. A family $\mathcal{F}$ of cts fcns $\Omega \to S$ is normal iff:
(i) $\mathcal{F}$ is equicts on every cpt set
(ii) $\forall z \in \Omega$, the values $\{f(z) \mid f \in \mathcal{F}\}$ is pre-cpt in S.

Take $S = \mathbb{C}$: $\mathcal{F}$ is normal iff:
(i) $\mathcal{F}$ is equicts on cpt sets
(ii) $\forall z \in \Omega$, the values $\{f(z) \mid f \in \mathcal{F}\}$ is bdd.
iff: (i) $\mathcal{F}$ is equicts on cpt sets
(ii) $\mathcal{F}$ is locally bdd.

Thm: A family $\mathcal{F}$ of analytic fcns is normal iff $\mathcal{F}$ is locally bdd.
Pf: We need to show locally bdd $\Rightarrow$ equicts on cpt sets.
Let C be the boundary of a closed disk of radius r in Ω centered at w_0. Let z, z_0 be 2 pts inside of C.
$$f(z) - f(z_0) = \frac{1}{2\pi i} \int_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right) f(\zeta)\, d\zeta$$

$= \frac{z-z_0}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta-z)(\zeta-z_0)} d\zeta$. Suppose $|f| \le M$ on C.
(M can be picked ind. of f, b/c f locally bdd & C cpt). Let z, z_0 be in the disk of radius $r/2$ centered at z_0.

$|f(z) - f(z_0)| \le \frac{4|z-z_0|}{2\pi r^2} \cdot 2\pi r M = \frac{4|z-z_0|}{r} \cdot M$
(all constants are ind of f)

$\Rightarrow \mathcal{F}$ equicts on disk of radius $r/2$.
(Every cpt set covered by finitely many disk $\Rightarrow$ equicts $\forall$ cpt sets)

Let $E \subseteq \Omega$ be cpt. Each pt $\zeta \in E$ is the center of a disk of radius r_ζ whose closure is in Ω. The disks centered at ζ of radius $r_\zeta/4$, $\zeta \in E$ is a cover of E. Pick a finite subcover.

$B(\zeta_1, r_1/4), \dots, B(\zeta_k, r_k/4)$

Suppose $\forall f \in \mathcal{F}$, $|f| \le M_\ell$ on $|z-\zeta_\ell| = r_\ell$. Let r be the smallest r_k & M the largest M_ℓ.
Fix $\varepsilon > 0$. Let $\delta = \min\{r/4, \varepsilon r/4M\}$. Let $|z-z_0| < \delta$, $z, z_0 \in E$. For some ℓ
$|z_0 - \zeta_\ell| < r_\ell/4 \Rightarrow |z-\zeta_\ell| < r_\ell/2$. We conclude
$|f(z) - f(z_0)| \le 4M_\ell/r_\ell |z-z_0| < \frac{4M}{r} \cdot \delta \le \varepsilon$. □

<u>Thm</u>: A locally bdd family of analytic fcns has locally bdd derivatives.

<u>Pf</u>: Fix a pt $w_0 \in \Omega$. Let C be the boundary of a closed disk of radius r centered at w_0.
For $|z - w_0| < r/2$, $f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta-z)^2} d\zeta$

$|f'(z)| \le \frac{1}{2\pi} \cdot M \cdot \frac{4}{r^2} \cdot 2\pi r = \frac{4M}{r}$, M ind. of f. □

Thm (Riemann Mapping Thm): Given any simply ctd region $\Omega \subsetneq \mathbb{C}$ (region = ctd open set) & given $z_0 \in \Omega$, $\exists !$ analytic $f(z)$ on Ω with $f(z_0)=0$, $f'(z_0)>0$, & f determines a 1-1 & onto map to $\{w \mid |w|<1\}$.
(ie bijective map from Ω into disk — its inverse is also analytic); (ie homeomorphism, which preserves simply ctd)

Pf: Outline: we consider the family $\mathcal{F}$ of all fcns g s.t. (i) g is analytic & injective on Ω.
(ii) $|g| \le 1$ on Ω.
(iii) $g(z_0)=0$ & $g'(z_0)>0$ (& real...)

Our desired fcn f is the one w/ the maximal $f'(z_0)$.

We will show: (1) $\mathcal{F} \neq \emptyset$
(2) $\exists f \in \mathcal{F}$ w/ a max'l deriv.
(3) f is the one we want (need onto).

(1): By assumption $\exists a \in \mathbb{C}$, $a \notin \Omega$. We can define $h(z)=\sqrt{z-a}$; ie, since Ω is simply ctd & $z-a \neq$ on Ω, there's $g(z)$ w/ $e^{g(z)} = z-a$. Set $h(z) = e^{\frac{1}{2}g(z)}$. This fcn is injective.
(If $h(z_1)=h(z_2) \Rightarrow h(z_1)^2 = h(z_2)^2 \Rightarrow z_1-a = z_2-a \Rightarrow z_1=z_2$). Moreover, it never attains a value & its negation, ie $h(z_1) \neq -h(z_2)$ (same argument, square both sides). The image of Ω under h (h takes open sets to open sets). This covers a disk $|w-h(z_0)|<\rho$. So the image of Ω under h never meets the disk $|w+h(z_0)|<\rho$. Ie, $|h(z)+h(z_0)| \ge \rho$ $\forall z \in \Omega$.
In particular $2|h(z_0)| \ge \rho$.

$g_0(z) = \frac{\rho}{4} \frac{|h'(z_0)|}{|h(z_0)|^2} \cdot \frac{h(z_0)}{h'(z_0)} \cdot \frac{h(z)-h(z_0)}{h(z)+h(z_0)}$. Claim $g_0 \in \mathcal{F}$.

We've composed $h(z)$ w/ a LFT, so $g_0(z)$ is injective.

$g_0(z_0) = 0$ (Note, denom $\neq 0$ b/c $\geq \rho$)

$g_0'(z_0) = \frac{\rho}{4} \cdot \frac{|h'(z_0)|}{|h(z_0)|^2} \cdot \frac{h(z_0)}{h'(z_0)} \cdot \frac{h'(z_0)}{2h(z_0)} > 0.$ (use prod. rule)

3/15 **Thm:** Given a simply cted region $\Omega \subsetneq \mathbb{C}$, & given $z_0 \in \Omega$, there exists a unique analytic fcn on Ω s.t. $f(z_0) = 0$, $f'(z_0) > 0$, & f defines a bijection $\Omega \to \{w \mid |w| < 1\}$

Pf: So far, defined $\mathcal{F}$ a family of fcns g s.t.

(i) g analytic & injective on Ω.

(ii) $|g(z)| \leq 1$ in Ω.

(iii) $g(z_0) = 0$, $g'(z_0) > 0$.

3 steps: (1) $\mathcal{F} \neq \emptyset$, (2) $\exists f \in \mathcal{F}$, $f'(z_0)$ maximal, (3) This f is surjective.

Started $\mathcal{F} \neq \emptyset$. Informally, forget about (3), (fix that up w/ a LFT). want $g: \Omega \to \mathbb{D}$ injective ($\mathbb{D}$ = unit disk).

If we can $g: \Omega \to \{w \mid |w - w_0| > \rho\}$ injective, then we can apply a LFT & turn the disk inside out, so we'd be done.

Take $a \notin \Omega$, set $h(z) = \sqrt{z-a}$, ie $(h(z))^2 = z - a$. If $h(z_1) = h(z_2)$ or $h(z_1) = -h(z_2)$, then $z_1 = z_2$. By the Open Mapping Thm, the image covers $|w - h(z_0)| < \rho$. The image does not meet $|w + h(z_0)| < \rho$.

⌈ halves the dist. from (0,0) & ½'s the ∢ to the real axis!

I.e., $|h(z) + h(z_0)| \geq \rho$ $\forall z$. In particular,

$2|h(z_0)| \geq \rho$.

$g(z) = \frac{\rho}{4} \cdot \frac{|h'(z_0)|}{|h(z_0)|^2} \cdot \frac{h(z_0)}{h'(z_0)} \cdot \frac{h(z) - h(z_0)}{h(z) + h(z_0)}$

Claim: $g \in \mathcal{F}$. g is injective: It is a LFT composed w/ h, which is injective. $g(z_0) = 0$ & $g'(z_0) > 0$.

$\left| \frac{h(z) - h(z_0)}{h(z) + h(z_0)} \right| = |h(z_0)| \cdot \left| \frac{1}{h(z_0)} - \frac{2}{h(z) + h(z_0)} \right| \leq |h(z_0)| \cdot \frac{4}{\rho}$ (from lower bds of $|h(z)|$ as above)

We see $|g(z)| \leq 1$.

(2): Let $B = \sup\{g'(z_0) \mid g \in \mathcal{F}\}$. Let $g_n \in \mathcal{F}$ be such that $g_n'(z_0) \xrightarrow{n\to\infty} B$. $\mathcal{F}$ is normal (b/c locally bdd; actually ≤ 1 on all of Ω). Let g_{n_k} be a subseq conv. unif. on cpt sets

$g_{n_k} \to f$, $|f(z)| \leq 1$, $f(z_0) = 0$, $f'(z_0) = B$.

Claim: $f \in \mathcal{F}$: NTS f is injective.

f is not constant, since $f'(z_0) = B > 0$. Let $z_1 \in \Omega$. WTS $f(z) - f(z_1) \neq 0$ on $\Omega \setminus \{z_1\}$. For $g \in \mathcal{F}$, set $g_1(z) = g(z) - g(z_1)$. g_1 is never zero on $\Omega \setminus \{z_1\}$. Any limit of these g_1's in the topology of unif. conv. on cpt sets must either be the constant fcn zero or never zero (Hurwitz's Thm). But $f(z) - f(z_1)$ is such a limit fcn, & $f'(z_0) \neq 0$, so $f(z) - f(z_1)$ is not constant, so $f(z) - f(z_1) \neq 0$ on $\Omega \setminus \{z_1\}$. We conclude f is injective, & $\therefore f \in \mathcal{F}$.

(3): NTS f is surjective.

Suppose $f(z) \neq w_0$ for some w_0 w/ $|w_0| < 1$.

Consider $l(z) = \frac{z - w_0}{1 - \overline{w_0} z}$. l takes disk to disk & w_0 to zero: $w_0 \to 0$ obvious.

$\underbrace{l(1) = \frac{1 - w_0}{1 - \overline{w_0}}, \quad l(-1) = \frac{-1 - w_0}{1 + \overline{w_0}}}_{\text{modulus } 1}$; $l(i) = \frac{i - w_0}{1 - \overline{w_0} i}$

$|l(i)|^2 = \left| \frac{i - \overline{w_0}}{1 - \overline{w_0} i} \cdot \frac{-i - \overline{w_0}}{1 + w_0 i} \right| = 1 \Rightarrow l$ takes $|z| = 1$ to $|z| = 1$

differing i or $-i$, so have same modulus.

So l either takes interior to interior or interior to exterior $\Rightarrow$? $l(w_0) = 0 \Rightarrow$ disk to disk

$l \circ f(z) = \frac{f(z) - w_0}{1 - \overline{w_0} f(z)}$ never 0 on Ω. Set

$F(z) = \sqrt{\frac{f(z) - w_0}{1 - \overline{w_0} f(z)}}$, ie $F(z)^2 = l \circ f(z)$.

F is injective, $|F| \leq 1$ ($F \notin \mathcal{F}$ b/c $F(z_0) \neq 0$, so apply a lft)

Set $G(z) = \frac{|F'(z_0)|}{F'(z_0)} \cdot \frac{F(z) - F(z_0)}{1 - \overline{F(z_0)} F(z)}$

$\uparrow$ makes $G' > 0$ $\quad \uparrow$ makes $z_0 \to 0$

$G(z_0) = 0.$

$$G'(z_0) = \frac{|F'(z_0)|}{1 - |F(z_0)|^2} \qquad \text{(product rule)}$$

$|G(z)| \le 1$, G is injective, so $G \in \mathcal{F}$

<u>Claim</u>: $|G'(z_0)| > f'(z_0) = B$

$$1 - |F(z_0)|^2 = 1 - \left| \frac{f(z_0) - w_0}{1 - \overline{w_0} f(z_0)} \right| = 1 - |w_0| \qquad (f(z_0) = 0)$$

$$\frac{d}{dz}\Big|_{z=z_0} \ell \circ f = \frac{(1 - \overline{w_0} f(z_0)) f'(z_0) - (f(z_0) + w_0)(-\overline{w_0} f'(z_0))}{(1 - \overline{w_0} \cancel{f(z_0)})^2}$$

$$= f'(z_0) - w_0(-\overline{w_0} f'(z_0)) = (1 + |w_0|^2) B$$

$$\frac{d}{dz} \operatorname{Log} \ell \circ f(z) = \frac{\frac{d}{dz} \ell \circ f}{\ell \circ f}$$

$$F(z) = e^{\frac{1}{2} \log \ell \circ f} \Rightarrow F'(z_0) = F(z_0) \cdot \frac{1}{2} \frac{(\ell \circ f)'(z_0)}{(\ell \circ f)(z_0)}$$

$\ell \circ f \big|_{z=z_0} = \frac{-w_0}{1} = -w_0$, so,

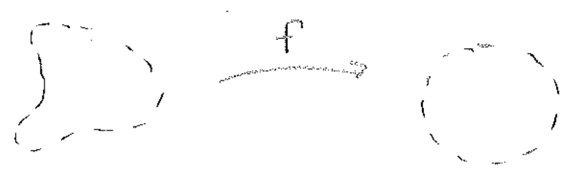
$$|F'(z_0)| = \overbrace{\sqrt{|w_0|}}^{|F(z_0)|} \cdot \frac{1}{2} \cdot \frac{(1 + |w_0|^2) B}{|w_0|} = \frac{1}{2} \cdot \frac{(1 + |w_0|^2) B}{\sqrt{|w_0|}}$$

$$|G'(z_0)| = \frac{\frac{\frac{1}{2}(1 + |w_0|^2) B}{\sqrt{|w_0|}}}{1 - |w_0|} = \frac{1}{2\sqrt{|w_0|}} \cdot (1 + |w_0|) B > B \quad ↯$$

We conclude $G \in \mathcal{F}$ & $|G'(z_0)| > f'(z_0)$ ↯ done

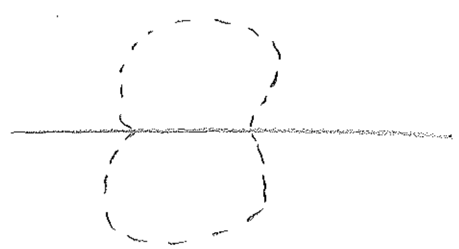
3/18 Prove uniqueness in RMT.
p. 184/1 ; p. 186/1,3 ; p. 193/1 due Friday.

Thm: Given any simply ctd region $\Omega \subsetneq \mathbb{C}$, $\exists!$ analytic $f(z)$ in Ω normalized by $f(z_0)=0$, $f'(z_0)>0$, s.t. f defines a bijective mapping of Ω onto the disk $\{w \mid |w|<1\}$.



Q: What happens as you approach the boundary?

Schwartz Reflection Principle (from midterm)



$F(z) = \overline{f(\bar{z})}$

(bijective, holomorphic)
Let's suppose f defines a conformal mapping from Ω onto another region Ω'. What happens to $f(z)$ as $z \to \partial\Omega$.

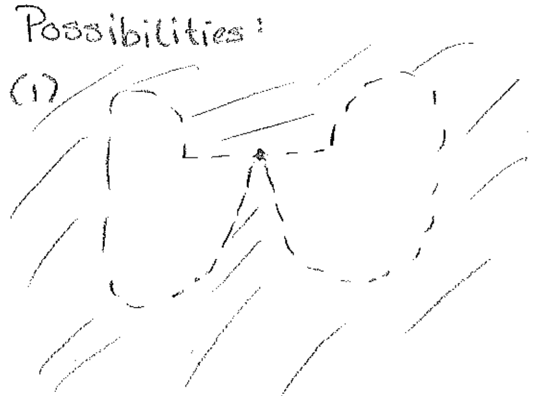
[Aside; suppose $f(0)=0$ & $f'(0)=0$. Then lowest order term of Taylor series is ≥ 2, so near 0, not injective $\Rightarrow$ it's ~ 2 to 1 (or 3 to 1, depending on the lowest order term)]

Def: For a sequence $z_n \in \Omega$, we say z_n approaches $\partial\Omega$ if $\forall z \in \Omega$ $\exists \varepsilon>0$ & N s.t. $n>N \Rightarrow |z_n - z| \geq \varepsilon$.
We say an arc $z(t)$ approaches $\partial\Omega$ if $\forall z \in \Omega$, $\exists \varepsilon>0$, t_0 s.t. $t>t_0 \Rightarrow |z(t)-z| \geq \varepsilon$

The disks of radius $\varepsilon(z)$ centered at z form a cover of Ω. Let $K \subset\subset \Omega$ be cpt. From this cover pick a finite subcover of K. The sequence z_n is eventually outside of K (b/c eventually outside each disk). Sim., the arc $z(t)$ is eventually outside of K.
Conversely, if the sequence is eventually outside any cpt subset, then $z_n \to \partial\Omega$.
For $z \in \Omega$, take $\varepsilon > 0$ s.t. $\{w \mid |w - z| \le \varepsilon\} \subseteq \Omega$.
(Sim. for arcs).
Approaching $\partial\Omega \Leftrightarrow$ being outside of any cpt set, eventually.

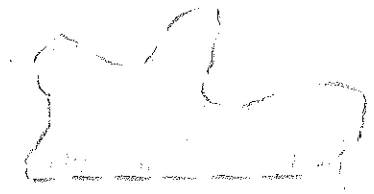
<u>Lemma</u>: Let f be a homeomorphism of a region Ω onto a region Ω'. If $\{z_n\}$ (or $z(t)$) approaches $\partial\Omega$, then $f(z_n)$ (or $f(z(t))$) approach $\partial\Omega'$.
<u>Pf</u>: Let $K \subset\subset \Omega'$ be cpt. $f^{-1}(K)$ is cpt. If $\{z_n\}$ (or $z(t)$) converges to $\partial\Omega$, it is eventually outside of $f^{-1}(K)$. Thus $f(z_n)$ is eventually outside of K □.
(or $f(z(t))$)

Suppose $f: \Omega \to \{w \mid |w| < 1\}$ is a conformal equivalence. Let's assume $f(z_0) = 0$. Suppose $\partial\Omega$ contains a seg. γ of a straight line. Rotate & translate so that γ lies on the real axis. Let this segment be $a < x < b$.
Possibilities:

(1)


(2)


reject both (1) & (2) w/ following assumptions.

We want: (3)  , ie, nothing immediately below seg is in Ω.

We assume every pt in γ has a nbhd whose intersection w/ $\partial\Omega$ is the same as its intersection w/ γ (rules out (1)). We call γ a <u>free boundary arc</u>. Every pt on γ is the center of a disk whose intersection with the $\partial\Omega$ is its diameter along $\mathbb{R}$-axis. Each half of the disks determined by this is either entirely in or entirely out of Ω, & at least one is in. If only one is in, we call it a <u>one-sided boundary pt</u>. If both are in, we call it a <u>2-sided boundary pt</u>.

<u>Thm</u>: Suppose the boundary of a s.c. region Ω contains a line seg γ as a one-sided free boundary arc. Then the fcn $f(z)$ which maps Ω onto $\{w \mid |w|<1\}$ can be extended to a fcn which is analytic & 1-1 on $\Omega \cup \gamma$. The image of γ will be an arc on the unit circle.

<u>Pf</u>: Consider a disk centered around $x_0 \in \gamma$ so small that the 1/2 of the disk which is in Ω does not contain z_0 (w/ $f(z_0)=0$). We can find a logarithm $\log f(z)$ on this half disk.

← want a little nbhd here where f analytic.
$\Rightarrow$ extend $\log f(z)$ into bottom 1/2 of disk, then exponentiate & piece together.
$\Rightarrow$ $\log f(z) = ai$ on seg, Schwartz Refl. principle, reflect.

$\operatorname{Log} f(z) = \log|f(z)| + i\arg(f(z))$. As $z \to \partial\Omega$, $\log f(z) \to$ purely imaginary value (b/c $|f(z)| \to 1$). By Schwartz's Reflection Principle, $\log f(z)$ has an analytic extension to the whole disk, & $\therefore$ $f(z)$ is analytic at x_0.
(If 2 disks overlap, they overlap on a part of $\mathbb{R}$-axis which contains a limit pt, so f is uniquely determined)
The extension on overlapping disks must coincide & determine an analytic fcn on $\Omega \cup \gamma$. $\square$

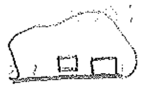
<u>Analytic Arcs</u>: A real or complex fcn $\varphi(t)$ of a real variable, t, defined on $a<t<b$ is said to be <u>real analytic</u> if $\forall\, t_0 \in (a,b)$,

$$\varphi(t) = \varphi(t_0) + \varphi'(t_0)(t-t_0) + \frac{\varphi''(t_0)}{2}(t-t_0)^2 + \cdots,$$

where we assume this converges on some interval centered at t_0: $(t_0-\rho, t_0+\rho)$
[ie fcn is given by a power series in its real variable].
This converges for $|t-t_0|<\rho$ where $t \in \mathbb{C}$.
[ie is strip around interval where fcn is holomorphic]

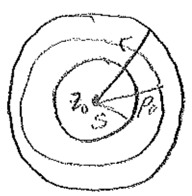
3/20 ② Analytic is a local property — can look at any closed curve in a small ball near a pt. But it doesn't need to be <u>any</u> curve — can be △ or □ for Morera's thm. — this allows us to reduce to a s. ctd. region.

∥ to axes.



· rect that hits I on bottom is a limit of rects that almost touch bottom
– DCT allows us to switch lim & $\int$
so $\int = 0$.

⑤ (a)



$\rho_0 = \frac{r+s}{2}$. For $\rho_0 \leq \rho \leq r$ & $|z - z_0| < s$,

$f(z)^2 = \frac{1}{2\pi i} \int_{|\zeta - z| = \rho} \frac{f(\zeta)^2}{\zeta - z} d\zeta$ (Cauchy's thm)

$|f(z)|^2 \leq \frac{1}{2\pi} \int_{|\zeta - z| = \rho} \frac{|f(\zeta)|^2}{|\zeta - z|} d\zeta$

$\leq C_{r,s} \int_{|\zeta - z_0|} |f(\zeta)|^2 d\zeta$

from $|\zeta - z|$

· average over all values of ρ btwn ρ_0 & r.

$\tilde{C}_{r,s} |f(z)|^2 = \int_{\rho_0}^{r} |f(z)|^2 d\rho \leq C_{r,s} \int_{\rho_0}^{\rho} \int_{|\zeta - z_0| = \rho} |f(\zeta)|^2 |d\zeta| d\rho$ $(r - \rho_0)$

$\leq C_{r,s} \int_0^{\rho} \int_{|\zeta - z_0| = \rho} |f(\zeta)|^2 d\zeta d\rho$

← = integral in polar coords usually $\rho d\rho$, but ρ is in $|d\zeta|$

$= C_{r,s} \int |f(z)|^2 dx dy$

(Could replace ^{2} by ^{1} ⇒ so actually bdd by L^1 norm on finite meas. sp's L^1 norm $\leq L^2$ norm (by Holders ineq.)

· conv. in any L^p ⇒ unif. conv. on cpt. sets.

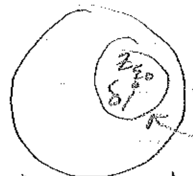
④ $f_\varepsilon(z) = f(z) + \varepsilon g(z)$ $f(0) = 0$ (simple) & no other zeros in $|z| \leq 1$
f, g holo on cl. ball.

(Rouche's Thm)

$|f(z) - f_\varepsilon(z)| = \varepsilon |g(z)| < |f(z)|$ max of g < min f

⇒ f_ε & f have same # of zeros. Let z_ε = ! zero.

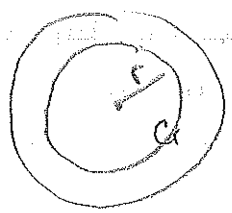
WTS for a fixed ε_0, if $\varepsilon-\varepsilon_0$ is small enough, z_ε close to z_{ε_0}.



want z_ε in this ball.

f_{ε_0} has one zero in this ball, by same pf, for $|\varepsilon-\varepsilon_0|$ small, f_ε has one zero in this ball (check f_ε not zero near ball — not b/c f_{ε_0} bdd away from zero & just change ε slightly)

(6)



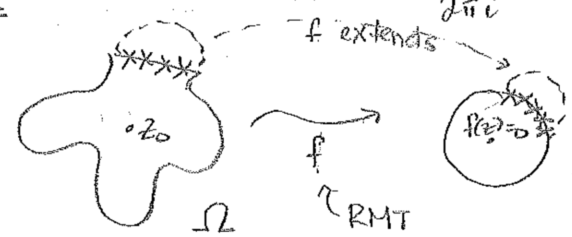
$\int_{C_r} f(\zeta)\,d\zeta = 0$ (Cauchy) take $\lim_{r\to 1}$ (DCT)

$\downarrow \qquad\qquad \downarrow$

$\int_{C_r} \frac{1}{z}\,dz \quad \neq \quad 0 \quad$ ↯.

$\quad \overset{\shortparallel}{2\pi i}$

=



A fcn $\phi:(a,b)\to\mathbb{C}$ is said to be real-analytic if $\forall t_0\in(a,b)$ $\exists \rho>0$ s.t. $\forall t\in(t_0-\rho,t_0+\rho)$, $\phi(t)=\phi(t_0)+\phi'(t_0)(t-t_0)+\cdots$

(ie power series converges)

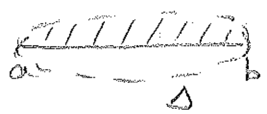
This converges for $|t-t_0|<\rho$, $t\in\mathbb{C}$. (what it conv. to is given by a power series & $\therefore$ analytic)

agree on line seg, which contains limit pt $\Rightarrow$ 2 ext'ns same

get $\mathbb{C}$-analytic fcn on a nbhd.

We say ϕ determines an analytic arc. It's called regular if $\phi'(t)\neq 0$ & simple if ϕ injective (ie, doesn't intersect itself).

We assume $\partial\Omega$ contains a regular, simple analytic arc γ. We assume $\exists$ a region Δ symmetric wrt (a,b)



w/ the prop. that $\phi(t) \in \Omega$ for t in the upper ½ of Δ & $\phi(t) \notin \Omega$ for t in lower half (sim. to one-sided boundary arc)

If necessary, shrink Δ so that $\phi(t)$ is never z_0 on Δ.

$f(\phi(t))$ defined only on upper ½ of Δ.

$\log(f(\phi(t)))$ by the reflection principle extends from the upper ½ of Δ to lower ½ (as before). Exponentiate & get $f\circ\phi$ extends from upper ½ to lower ½, as well.

b/c $\phi' \neq 0$ in Δ, ϕ is locally injective: $\forall t \in (a,b)$ $\phi'(t) \neq 0$. So if $t_0 \in (a,b)$ $\exists \rho > 0$ s.t. $\phi|_{B(t_0,\rho)}$ is injective. $f\circ\phi|_{B(t_0,\rho)}$. Since ϕ^{-1} is holo,

f is holo. on $\phi(B(t_0,\rho))$. Similarly, put all balls together, & f extends on Δ.

<u>Weierstrass Approx Thm</u>: If f is cts on a cpt interval, then f can be uniformly approx by polynomials. (1-dim.)

In $\mathbb{C}$, same is true for polys in z and $\bar{z}$.

(but for holo. fcns, want to only have 1 var, z)

ie holo on nbhd containing disk

If f is holo. on a cl. disk $\overline{B(z_0,r)}$, then $f(z) = C_n(z-z_0)^n$ on the ball.

↑ conv. unif on cpt sets inside region of holo, this ball is cpt $\Rightarrow$ conv. unif. on $\overline{B(z_0,r)}$ to f.

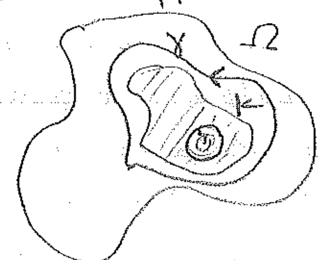
$\frac{1}{z}$ on $\{|z|=1\}$. If $P_n(z) \to 1/z$, then $\underbrace{\int_{|z|=1} 1/z \, dz}_{2\pi i} = \lim_{n\to\infty} \int_{|z|=1} P_n(z) = 0$ ↯

If complement of cpt set is not connected, cannot approx all holo. fcns by polys:

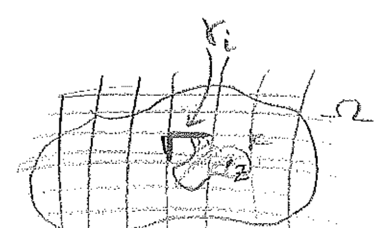
$\frac{1}{z-z_0}$ cannot be approx.

3/22 <u>Runge's approximation Thm</u>: Any fcn holo. in a nbhd of a cpt set K can be approx. uniformly on K by rat'l fcns whose singularities are in K^c. If K^c is cnd, any fcn holo. on a nbhd of K can be approx. unif. on K by polynomials.

<u>Lemma</u>: Suppose f is holo. in an open set Ω & $K \subset\subset \Omega$. Then $\exists$ finitely many line segs, $\gamma_1, \dots, \gamma_N$ in $\Omega \setminus K$ s.t. $f(z) = \sum_{n=1}^{N} \frac{1}{2\pi i} \int_{\gamma_n} \frac{f(\zeta)}{\zeta - z} d\zeta$, $\forall z \in K$



($f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$)



<u>Pf</u>: Let $d = c \cdot \mathrm{dist}(K, \Omega^c)$, $c < \frac{1}{\sqrt{2}}$.
Consider a grid formed by closed squares with sides $\parallel$ to the axes & side length d. Let $\mathcal{Q} = \{Q_1, \dots, Q_M\}$ be the finite collection of squares which intersect K. Give ∂Q_j the counterclockwise orientation.
Let $\gamma_1, \dots, \gamma_N$ denote the sides of squares in $\mathcal{Q}$ that do not belong to 2 adj. squares in $\mathcal{Q}$.
Each γ_n is in Ω b/c of the choice of d.
Each γ_n does not intersect K, by our choice of γ_n. Take a pt $z \in K$, z not on a boundary of Q_j. $\exists j$ w/ $z \in \mathring{Q}_j$ (interior of Q_j). By Cauchy's formula, $\frac{1}{2\pi i} \int_{\partial Q_m} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z), & m = j \\ 0, & m \neq j \end{cases}$

For all such z, $f(z) = \sum_{m=1}^{M} \frac{1}{2\pi i} \int_{\partial Q_m} \frac{f(\zeta)}{\zeta - z} d\zeta$.
- If 2 squares share a side, they cancel out in the integral, and so, for such z, $f(z) = \sum_{j=1}^{N} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(\zeta)}{\zeta - z} d\zeta$. By continuity, this formula holds for all $z \in K$.

Suffices to approx each $\int_{\gamma_j} \frac{f(\zeta)}{\zeta - \bar{j}} d\zeta$ by rat'l fcns.

true for piecewise smooth curves

<u>Lemma</u>: For any line seg contained entirely within $\Omega \setminus K$ $\exists$ a seq. of rat'l fcns w/ singularities on γ that <u>approx.</u> $\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$.

ie, converge unif. on K

<u>Pf</u>: Let $\gamma(t) : [0,1] \to \mathbb{C}$ be a parametrization of γ.

$\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t)\, dt$ call this $\int_0^1 F(t,z)\, dt$

$F(t,z)$ is a continuous fcn ($F : [0,1] \times K \to \mathbb{C}$) (note denom $\neq 0$ b/c $\gamma \cap K = \emptyset$) F is unif cts b/c $[0,1] \times K$ is cpt. The Riemann sums defining $\int_0^1 F(z,t)\, dt$ converge to $\int_0^1 F(t,z)dt$ unif. for $z \in K$ (b/c F unif. cts in z,t). The Riemann sums are

$$\sum_{k=1}^{n} F\left(\tfrac{k}{n}, z\right) \cdot \frac{1}{n} = \sum_{k=1}^{N} \frac{f(\gamma(k/n))}{\gamma(\frac{k}{n}) - z} \cdot \gamma'\left(\tfrac{k}{n}\right) \cdot \frac{1}{n}$$

(a sum of rat'l fcns – variable is z)

$$\longrightarrow \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \text{ unif. on } K$$

The next lemma finishes the pf of Runge's Thm.

<u>Lemma</u>: If K^c is ctd, & $z_0 \notin K$, then the fcn $\frac{1}{z - z_0}$ can be approx. unif. on K by polynomials.

<u>Pf</u>: First choose a pt z_1 that is outside a large open disk D centered at 0 which contains K. For $z \in K$,

$$\frac{1}{z - z_1} = \frac{-1}{z_1} \cdot \frac{1}{1 - (z/z_1)} = \sum_{n=1}^{\infty} \frac{-z^n}{z_1^{n+1}}.$$ This conv. unif. on K.

$\frac{1}{(z - z_1)^k}$ can be approx. unif on K for any k by polys. It suffices to show $\frac{1}{z - z_0}$ can be approx. uniformly by polys in $\frac{1}{z - z_1}$. Using K^c ctd, let γ be a curve from z_0 to z_1.

Parameterize γ by $\gamma(t) : [0,1] \to K^c$; $\gamma(0) = z_0$, $\gamma(1) = z_1$

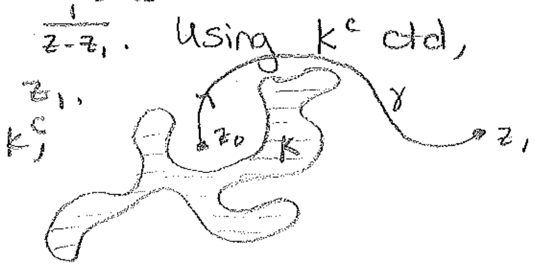


Image of γ is a cpt set, So γ & K have a non-zero dist btwn them. Let $\rho = \frac{1}{2}\,\text{dist}(K,\gamma) > 0$.
Choose a sequence of pts $\{w_0, \dots, w_\ell\}$ on γ w/ $w_0 = z_0$, $w_\ell = z_1$, $|w_j - w_{j-1}| < \rho$. We claim if w is a pt on γ & w' is any other pt w/ $|w - w'| < \rho$, then $\frac{1}{z-w}$ can be approx by polys in $\frac{1}{z-w'}$ unif on K.

$$\frac{1}{z-w} = \frac{1}{z-w'} \cdot \frac{1}{1 - \underbrace{\frac{w-w'}{z-w'}}_{<1/2 \;\Rightarrow\; \frac{<\rho}{2\rho} \leftarrow z\in K,\, w' \in \gamma}} = \sum_{n=0}^{\infty} \frac{(w-w')^n}{(z-w')^{n+1}},$$
& this converges unif. for $z \in K$.

This completes the proof $\square$

4/1 Gamma fcn & Zeta Fcn.

$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\,dt$, $Re(s) > 0$ $\quad s = \sigma + i\tau$, $t^{s+i\tau} = t^s e^{i\tau \log t}$

$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$, $Re(s) > 1$

In these regions, these fcns are analytic fcns of parameter s. Pf (of $\Gamma(s)$)

① Diff quotient: WTS: $\frac{\Gamma(s+h) - \Gamma(s)}{h} \xrightarrow{h \to 0} L(s)$ where

$L(s) = \int_0^\infty (\ln t)\, t^{s-1} e^{-t}\,dt$ (still conv. for $Re(s) > 0$) (differentiate under the $\int$)

$\Gamma(s+h) - \Gamma(s) - hL(s) = \int_0^\infty e^{-t} \frac{1}{t}\left[t^{s+h} - t^s - h(\ln t)\, t^s\right] dt$, $Re(s) > 0$

say $h < Re(s)$, then in $[\cdots]$ is diff. quotient of $t^s = (t^s)'$:

$[\cdots] = h^2 \int_0^1 (1-u)\, t^{s+uh} (\ln t)^2\,du$ $\leftarrow$ taylor's formula:

(b/c $g(s+h) - g(s) - g'(s)h = \int_0^1 (1-u) g''(s+uh)\,du \cdot h^2$

$|\Gamma(s+h) - \Gamma(s) - hL(s)| \le \int_0^\infty e^{-t}\, \underbrace{t^{Re(s)-1} (\ln t)^2 (h^2)\, |t^{uh}|}$

if $|h| > Re(s)/2$ $\longrightarrow$ $\le h^2 (\ln t)^2\, t^{Re\, s/2 - 1}$ conv. ✓

② Apply Morera's Thm - need to check cts fcn, apply Fubini.

Pf (of $\zeta(s)$): ① Same as above
② k^{-s} unif. conv. when $Re(s)>1$, so limit is holo.

Γ & ζ can be analytically continued to larger regions.
Γ is interesting b/c $\Gamma(n)=(n-1)!$
$\Gamma(1)=\int_0^\infty e^{-t}dt=1$
By integration by parts (for $Re(s)>0$),
$\Gamma(s)=\int_0^\infty \frac{d}{dt}\left(\frac{t^s}{s}\right)e^{-t}dt=\int_0^\infty \frac{t^s}{s}e^{-t}dt=\frac{1}{s}\Gamma(s+1) \Rightarrow !$
boundary terms $=0$
ie, Γ interpolates the factorial.

Analytic Continuation: $\Gamma(s)$ is analytic in $\{Re\, s>0\}$,
$\frac{\Gamma(s+1)}{s}$ is analytic in $\{Re\, s>-1\}\setminus\{0\}$.
Repeated application (together w/ uniqueness thm of analytic fcns): Γ can be continued to an analytic fcn in $\mathbb{C}\setminus\{0,-1,-2,\dots\}$ with simple poles at $\{0,-1,\dots\}$.

Thm: ζ can be continued to an analytic fcn in $\mathbb{C}\setminus\{1\}$, with a simple pole at $s=1$, so $\zeta(s)-\frac{1}{s-1}$ is analytic in $\mathbb{C}$.
Pf: Try to extend to $Re(s)>0$, $s\neq 1$, as a first step.
Idea: compare Σ to $\int_1^\infty x^{-s}\,dx$ (anal. fcn of s for $Re(s)>1$)
$=\frac{1}{s-1}$ ← anal in $\mathbb{C}\setminus\{1\}$.
Show $\int-\Sigma$ extends analytically a little bit.
$$\zeta(s)-\frac{1}{s-1}=\sum_{n=1}^{\infty}\underbrace{\int_n^{n+1}}_{\text{full integral after sum}}\Big[\underbrace{\frac{1}{n^s}}_{}-\frac{1}{x^s}\Big]\,dx \qquad x^s-n^s=(x-n)\,O(n^{Re(s)-1})$$
$$=\sum_{n=1}^{\infty}\int_n^{n+1}\underbrace{\frac{x^s-n^s}{n^s x^s}}_{\text{integrand is } O(n^{-Re(s)-1})}\,dx$$
Conv. unif. on cpt sets where $\{Re(s)\geq\varepsilon>0\}$
better by $1/n$ by terms above

so limit is analytic in larger region.

Further ext'n : $f(x) = x^{-s}$ (f depends on s analytically)

WTS $\zeta(s) - \frac{1}{s-1}$ as before is entire. Expand around $n = x$:

$$n^{-s} - x^{-s} = \sum_{k=1}^{M} \underbrace{\frac{f^{(k)}(x)}{k!}}_{\text{depends on } s \text{ analytically}} (n-x)^k + \underbrace{\int_0^1 (1-t)^M f^{(M+1)}(x + t(n-x))\,dt\,\frac{(n-x)^{M+1}}{(M+1)!}}_{\text{error term} = G_M(x,s)}$$

depends on s anal.

$f^{(M+1)}(x) = \frac{1}{x^{s+M+1}}(\cdots)$ ← use similar argument as above:

Then $\sum_{n=1}^{\infty} \int_n^{n+1} G_M(x,s)\,dx$ is an an analytic fcn when it conv. unif., when $Re(s) > -M$

(b/c when $Re(s) + M + 1 > 1$)

Main terms: They are a linear comb. of terms of the form $(-s-k+1)\int_1^{\infty} x^{-s-k} \underbrace{\{x\}^k}_{\text{from } (n-x)^k \text{ in integral } \int_n^{n+1}} dx$ $\{x\}$ = fractional part of x.

from $f^{(k)}(x)$

1-periodic fcn

$a(x)$

$T_s a = (1-s)\int_1^{\infty} x^{-s} a(x)\,dx$ ($a(x)$ bdd meas. 1 periodic)

Then T_s is analytic in s in $\mathbb{C}$.

If $s = s+k$, then get above expression.

* int of 1-periodic fcn is 1-per. when $\int_0^1 s(x)\,dx = 0$ (ie. mean value $= 0$). On each $\int_n^{n+1}$, subtract $\int_0^1 s(x)\,dx$, which is computable. The $(1-s)$ ensures no further poles are created.

4/3 $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$, $Re(s) > 1 \Rightarrow |p^{-s}| < 1$

↑ sum of geom. series

Heuristic:

$$RHS = \prod_p \sum_{n=0}^{\infty} (p^{-ns}) = \prod_p (1 + p^{-s} + p^{-2s} + \dots)$$

$$= 1 + \sum_n \sum_{\substack{p_1, \dots, p_n \\ k_1, \dots, k_n}} (p_1^{k_1} \cdots p_n^{k_n})^{-s} = \sum_{n=1}^{\infty} n^{-s}$$

Fundamental Thm of Arith → check series conv. abs.

To make rigorous, need to deal w/ errors.

Conv. of RHS ✓ & LHS ✓

So can approx LHS by finite sum & RHS by finite prod.

Analytic Continuation of $\zeta(s)$:

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx$$

$$n^{-s} - x^{-s} = \sum_{k=0}^{M} \frac{\partial^k x (n^{-s} - x^{-s})}{k!} + \underbrace{E_{n,m}(x,s)}_{\text{error - ok on } Re(s) > -M}$$

Need to check main terms:

Need $\sum_{n=1}^{\infty} \int_n^{n+1} (-s-k+1) x^{-s-k} \{x\}^k \, dx$

$= \int_1^{\infty} (-s-k-1) x^{-s-k} \{x\}^k dx$ to be entire

$= T_{s+k}(\{x\}^k)$ ← translate

↳ bdd meas. 1-periodic

$T_s(a) = \int_1^{\infty} (-s-1) x^{-s} a(x) \, dx$ → analytic on $Re(s) > 1$ ↙ where $\int$ conv. unif.

- want to use by parts, but $\int a(x)$ not 1-periodic.

Trick: $\mathcal{L}(a)(x) = \int_0^x a(x) - x \underbrace{\int_0^1 a(x) dx}_{\mathbb{E}(a) \text{ - expectation of } a}$

$\mathcal{L}(a)'(x) = a(x) - \mathbb{E}(a)$

So $T_s(a) = \int_1^{\infty} -(s+1) x^{-s} \mathcal{L}(a)'(x) \, dx + \mathbb{E}(a)$

$= \underbrace{\int_1^{\infty} -s(s+1) x^{-(s+1)} \mathcal{L}(a)(x) \, dx}_{\text{analytic on } Re(s) > 0.} + \text{ok}$

Repeat → $\mathcal{L}(a)(x) = a_1$ $\quad = s T_{s+1}(a_1)$

In general, $T_s(a) = \underbrace{s(s+1)\cdots(s+k-1)\, T_{s+k}(a_k)}_{\text{analytic on } Re(s) > -k} + \text{ok stuff}$

So $T_s(a)$ analytic on $\mathbb{C}$

(bdary terms - if $Re(s)>1$, terms are 0)

An Identity: $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$ (Recall: $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$)

Pf: $\Gamma(1-s) = \int_0^\infty e^{-v} v^{-s} dv$

$= t\int_0^\infty e^{-tv}(tv)^{-s} dv$

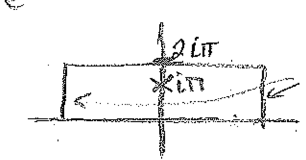
$$\Gamma(s)\Gamma(1-s) = \int_0^\infty e^{-t} t^{s-1} \int_0^\infty e^{-v} v^{-s} dv dt$$
$$= \int_0^\infty e^{-t} t^{s-1}\, t\int_0^\infty e^{-tv}(tv)^{-s} dv dt$$
$$\overset{\text{Fubini}}{=} \int_0^\infty \int_0^\infty e^{-t(1+v)} v^{-s} dt dv$$
$$= \int_0^\infty \frac{v^{-s}}{1+v} dv$$

Suffices to show identity for s real, $s=a$, $0<a<1$.

$\Gamma(a)\Gamma(1-a) = \int_0^\infty \frac{v^{-a}}{1+v} dv$ $\quad v = e^t$

$= \int_{-\infty}^\infty \frac{e^{-at}}{1+e^t} e^t dt$ (contour integral)

$= \frac{\pi}{\sin \pi a}$ $\quad \frac{e^{(1-a)z}}{1+e^z}$ has a pole at $z=(2k+1)i\pi$



$\to 0$ b/c real parts $\to \infty$

Other identities of $\zeta(s)$:

$\Theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$ (discrete version of Gaussian)

Next time we'll show: $\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty u^{s/2-1}(\Theta(u)-1)\,du$

4/5

<u>Theta Function</u>

$\Theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$, $t > 0$ conv.

- Poisson summation $\Rightarrow \Theta(t) = t^{-1/2}\,\Theta(t^{-1})$
- $t > 1 \Rightarrow$ fast decay $\Rightarrow$ unif. conv.
- $0 < t < 1 \Rightarrow \Theta(t) = O(t^{-1/2})$, since $t^{-1} > 1$ &

<u>Xi Function</u>

$\xi(s) = \pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$

<u>Thm</u>: For $Re(s) > 1$, $\xi(s) = \frac{1}{2}\int_0^\infty u^{s/2-1}(\Theta(u)-1)\,du$.

<u>Pf</u>: $\frac{1}{2}(\Theta(u)-1) = \sum_{n=1}^{\infty} e^{-\pi n^2 u}$ b/c Θ even & subtracting off 0^{th} term.

- can interchange sum & $\int$ b/c of DCT & $\Theta(t) = O(t^{-1/2})$.

$\int_0^\infty u^{s/2-1} e^{-\pi n^2 u}\,du = \frac{1}{\pi n^2}\int_0^\infty \left(\frac{t}{\pi n^2}\right)^{s/2-1} e^{-t}\,dt$

$= \frac{1}{(\pi n^2)^{s/2}}\,\Gamma(s/2)$

By DCT & $\Theta(t) = O(t^{-1/2})$, sum comes outside of the $\int$ & get $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

<u>Thm</u>: ξ is holomorphic on $Re(s) > 1$, meromorphic on $\mathbb{C}$, with simple poles at 0 & 1, & $\xi(s) = \xi(1-s)$ $\forall s \in \mathbb{C}$.

<u>Pf</u>: By def, ξ is mero on $\mathbb{C}$ w/ possible poles at 1 (ζ), 0 (Γ), $-2, -4, \ldots$

For $Re(s) > 1$, $\xi(s) = \frac{1}{2}\int_0^\infty u^{s/2-1}(\Theta(u)-1)\,du$

$= \frac{1}{2}\int_1^\infty u^{s/2-1}(\Theta(u)-1)\,du + \frac{1}{2}\int_0^1 u^{s/2-1}(u^{-1/2}\,\Theta(u^{-1})-1)\,du$

$= \frac{1}{2}\int_1^\infty u^{s/2-1}(\Theta(u)-1)\,du + \frac{1}{2}\int_0^1 u^{\frac{s-1}{2}-1}(\Theta(u^{-1})-1)\,du + \frac{1}{2}\int_0^1 u^{\frac{s-3}{2}} - u^{\frac{s-2}{2}}\,du$

$t = u^{-1}$

$= \underbrace{\frac{1}{2}\int_1^\infty u^{s/2-1}(\Theta(u)-1) + u^{\frac{1-s}{2}-1}(\Theta(u)-1)\,du}_{}+ \frac{1}{s-1} - \frac{1}{s}$

Conv. locally unif on s b/c of fast decay of $\Theta(u)-1$
$\Rightarrow$ analytic.

By analytic continuation $\xi(s) =$ $\forall s \in \mathbb{C}$
$\Rightarrow$ thm.

□

* If Re(s) huge, $\zeta(s) \neq 0$ b/c 1st term = 1 & rest won't sum to smth > 1 ...

<u>Thm</u>: The <u>only</u> zeros of $\zeta(s)$ outside $0 \le Re(s) \le 1$ lie at $-2, -4, -6, \dots$

<u>Pf</u>: If $Re(s) > 1$, $\zeta(s) = \prod_p \frac{1}{1+p^{-s}} \neq 0$, b/c $\left|\log \prod_p \frac{1}{1+p^{-s}}\right| < \infty$.

For $Re(s) < 0$, use $\zeta(s) \to \zeta(1-s)$ & $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{1-s/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$\Rightarrow \zeta(s) = \frac{\underbrace{\pi^{s-1/2}}_{\text{no zeros}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}{\Gamma(s/2)}$$

$\zeta(1-s)$: if $Re(s) < 0$, $Re(1-s) > 1$, non-zero

poles of $\Gamma(s/2)$: $0, -2, -4, -6, \dots$

zeros of $\Gamma\left(\frac{1-s}{2}\right)$: none by following lemma. □

<u>Lemma</u>: $\Gamma(s)$ has no zeros.

<u>Pf</u>: Last time showed $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$

RHS $\neq 0$ b/c sin is entire. So $\Gamma(s) = 0 \Leftrightarrow \Gamma(1-s) = \infty$
$\Leftrightarrow 1-s \in \{0, -1, -2, \dots\} \Leftrightarrow s \in \{1, 2, \dots\} \Rightarrow \Gamma(s) \neq 0$ (b/c $\Gamma(s)$ is a factorial) □

<u>Q</u>: Where are the zeros inside $0 \le Re(s) \le 1$.

<u>Riemann Hypothesis</u>: The only zeros lie at $Re(s) = 1/2$.

<u>Pf</u>: Open.

<u>Thm</u>: There are no zeros of $\zeta(s)$ along $Re(s) = 1$.

(<u>Note</u>: $\Rightarrow$ No zeros along $Re(s) = 0$)

<u>Pf</u>: Idea: Look at $\log|\zeta(\sigma + it)|$ for $1 < \sigma < 1+\delta$.

- If $Re(s) > 1$, $\log(\zeta(s)) = \sum_p \text{Log}\left(\frac{1}{1+p^{-s}}\right)$ ← use Taylor series

$$= \sum_p \sum_m \frac{p^{-ms}}{m} = \sum_n c_n n^{-s} \quad \text{where } c_n = \begin{cases} 1/m & \text{if } n = p^m \\ 0 & \text{otherwise.} \end{cases}$$

$$\log|\zeta(s)| = Re(\log(\zeta(s))) = \sum_n c_n Re(n^{-s}) \qquad n^{-s} = e^{-s\log n}$$
$$= \sum_n c_n n^{-Re(s)} \cos((\text{Im}\, s)\log n)$$

$$3 + 4\cos\theta + \cos 2\theta = 2 + 4\cos\theta + 2\cos^2\theta = 2(1+\cos\theta)^2 \ge 0$$

So if $\sigma > 1$, we have

$\log(|\zeta(\sigma)|^3 |\zeta(\sigma+it)|^4 |\zeta(\sigma+2it)|)$

$= \sum_n c_n n^{-\sigma} [\underbrace{3 + 4\cos(t\log n) + \cos(2t\log n)}_{\geq 0 \ \forall n}] \geq 0$

If $\zeta(1+it) = 0$ for some $t \neq 0$,

$\underbrace{\zeta^3(\sigma)}_{\substack{\text{pole of} \\ \text{order 3 at} \\ \sigma=1}} \underbrace{\zeta^4(\sigma+it)}_{\substack{\text{zero of} \\ \text{order at} \\ \text{least 4} \\ \text{at } \sigma=1}} \underbrace{\zeta(\sigma+2it)}_{\substack{\text{bounded near} \\ \sigma=1}} \Big\} \Rightarrow$ zero at $\sigma = 1$. & cts near $\sigma = 1$

But $\log 0 = -\infty$ ↯. $\square$

4/8 $\mathbb{R}$-analysis & $\mathbb{C}$-analysis.

<u>Fourier Transform</u>: If $f: \mathbb{R} \to \mathbb{C}$,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx \qquad (f \in L^1)$$

If f is "nice", $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$, the <u>inverse Fourier transform</u> (eg. when $|f(x)| \leq \frac{A}{1+x^2}$ & $|\hat{f}(\xi)| \leq \frac{A}{1+\xi^2}$)

- results about one will hold about other.

<u>Poisson summation formula</u>:

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

For each $a > 0$, let $\mathcal{F}_a$ be the class of fcns f s.t.

(1) f is holomorphic in the horiz. strip

$S_a = \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < a\}$

(2) $\exists A > 0$ s.t. $|f(x+iy)| \leq \frac{A}{1+x^2}$ $\forall x \in \mathbb{R}$ $|y| < a$

(ie, think of y as fixed)

$\mathcal{F} = \bigcup_{a>0} \mathcal{F}_a$.

<u>ex</u>: $f(z) = e^{-\pi z^2} = e^{-\pi(x+iy)^2} = e^{-\pi(x^2-y^2+2ixy)} \in \mathcal{F}_a \ \forall a > 0$

$f(z) = \frac{1}{c^2+z^2}$ simple poles at $z = \pm ci \in \mathcal{F}_a$ for $0 < a < c$.

If $f \in \mathcal{F}_a$, then $f^{(n)} \in \mathcal{F}_b$ $\forall$ $0 < b < a$.

<u>Thm</u>: If $f \in \mathfrak{F}_a$ for some $a>0$, then $|\hat{f}(\xi)| \le B e^{-2\pi b|\xi|}$ $\forall\ 0<b<a$ (B depends on b).

(the larger a is, the better the decay - if $a=0$, $\hat{f}$ just bdd, which is also from $f \in L^1$, so lose all info)

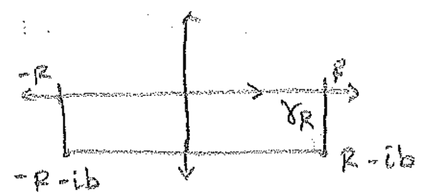
<u>Pf</u>: Recall $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$

When $b=0$, $|\hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(x)|\, dx \le C$, trivially true.

$\qquad\qquad\qquad\qquad\qquad\quad \uparrow \le A/1+x^2$

Suppose $0<b<a$, & assume $\xi>0$. Consider $g(z) = f(z) e^{-2\pi i z \xi}$

Contour:



$\int_{\gamma_R} g\, dz = 0$ by Cauchy's Thm.

<u>Claim</u>: as $R \to \infty$, the integral over the vertical sides of γ_R tend to 0.

<u>Pf</u>: $\left| \int_{-R-ib}^{-R} g(z)\, dz \right| \le \int_0^b \left| f(-R-it) e^{-2\pi i(-R-it)\xi} \right| d\xi$

$\leftarrow$ bdd for $|f| \le \frac{1}{1+x^2}$, R huge

$\le \int_0^b \frac{A}{R^2} \underbrace{e^{-2\pi t \xi}}_{<1}\, dt \le C \cdot \frac{1}{R^2} \to 0$ as $R \to \infty$

Similarly for other side.

Taking $R \to \infty$, we see $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x-ib) e^{-2\pi i(x-ib)\xi}\, dx$

(ie, two horiz. segs are =)

We conclude $|\hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(x-ib)| \left| e^{-2\pi i (x-ib)\xi} \right| dx$

$\le \int_{-\infty}^{\infty} \frac{A}{1+x^2} e^{-2\pi b \xi}\, dx \le C e^{-2\pi b \xi}$

If $\xi<0$, shift contour up. $\square$

Thm: If $f \in \mathcal{F}_a$ for some $a>0$, the Fourier inversion formula holds. $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi \quad \forall x \in \mathbb{R}$.

Show can move $\frac{\partial}{\partial \bar{\lambda}}$ inside

Lemma: If $A>0$ & $B \in \mathbb{R}$, then $\int_0^\infty e^{-(A+iB)\xi} \, d\xi = \frac{1}{A+iB}$.

Pf: $|e^{-(A+iB)\xi}| = e^{-A\xi}$

$\frac{\partial}{\partial \bar{\lambda}} F(\lambda) = \int_0^\infty \frac{\partial}{\partial \bar{\lambda}} e^{-\lambda \xi} \, d\xi = 0 \quad \to = 0$

$F(\lambda) = \int_0^\infty e^{-\lambda \xi} \, d\xi = 1/\lambda$ when $\lambda > 0$. F is holo. on $\mathrm{Re}(\lambda) > 0$ & agrees w/ $1/\lambda$ on $(0,\infty)$. So $F(\lambda) = 1/\lambda$ on $\mathrm{Re}(\lambda) > 0$ by uniqueness. ∎

F holo on $\mathrm{Re}\,\lambda > 0$:

Let γ be a small closed curve

$\int_\gamma \int_0^\infty e^{-\lambda \xi} \, d\xi \, d\lambda$

$\mathrm{Re}\,\lambda > 0$ dom

⇓ Fubini

$\int_0^\infty \int_\gamma e^{-\lambda \xi} \, d\lambda \, d\xi$

$= 0$

Pf of Thm: $\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi = \int_0^\infty \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi + \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi$.

Choose $0 < b < a$, $\xi > 0$

$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx = \int_{-\infty}^{\infty} f(u - ib) e^{-2\pi i (u - ib)\xi} \, du$

push contour down by same argument as before.

$\int_0^\infty \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi = \int_0^\infty \int_{-\infty}^{\infty} f(u - ib) e^{-2\pi i (u - ib)\xi} e^{2\pi i x \xi} \, du \, d\xi$

$= \int_{-\infty}^{\infty} f(u-ib) \int_0^\infty e^{-2\pi i (u - ib - x)\xi} \, d\xi \, du$

$= \int_{-\infty}^{\infty} f(u-ib) \frac{1}{2\pi b + 2\pi i (u - x)} \, du$

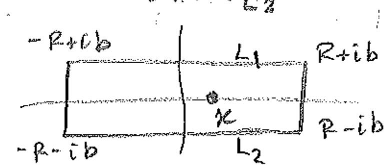
$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u-ib)}{u - x - ib} \, du$ ← param. of a line $\int$, so,

$= \frac{1}{2\pi i} \int_{L_1} \frac{f(\zeta)}{\zeta - x} \, d\zeta$, L_1 is the line $\{u - ib \mid u \in \mathbb{R}\}$, traversed from left to right

$\int_{-\infty}^0$ push contour up b/c $\xi < 0$:

$\int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi = -\frac{1}{2\pi i} \int_{L_2} \frac{f(\zeta)}{\zeta - x} \, d\zeta$, L_2 = real line shifted up by b left to right

$x \in \mathbb{R}$. γ_R:



$f(x) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(\zeta)}{\zeta - x} \, d\zeta$. Take $R \to \infty$, & vert. segs $\to 0$.

$\Rightarrow f(x) = \frac{1}{2\pi i} \left(\int_{L_1} \frac{f(\zeta)}{\zeta - x} \, d\zeta - \int_{L_2} \frac{f(\zeta)}{\zeta - x} \, d\zeta \right)$

$= \int_{-\infty}^{\infty} \hat{f}(\zeta) e^{2\pi i \zeta x} \, d\zeta$ ∎

4/10 $a>0$, $\mathcal{F}_a$ is those fcns f s.t.

① f is holo. on the strip $S_a = \{z\in\mathbb{C} \mid |\mathrm{Im}(z)|<a\}$

② $\exists$ a constant $A>0$ s.t. $|f(x+iy)| \le \frac{A}{1+x^2}$ (ie, fcn is L^1 on each copy of real line)

$\mathcal{F} = \bigcup_{a>0} \mathcal{F}_a$

$\forall f \in L^1(\mathbb{R})$, define $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi}\,dx$, $\xi\in\mathbb{R}$.

<u>Thm</u>: If $f\in\mathcal{F}$, then $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi}\,d\xi$.

Notice if $f_n \in L^1 \cap \mathcal{F}$, s.t. $|f_n| + |\hat{f}_n| \le g$, $g\in L^1$, & $f_n \to f$ a.e. (f may not be in $\mathcal{F}$, but will be cts - allowing a to shrink as n changes)

By DCT $\hat{f}_n \to \hat{f}$ everywhere

$f_n(x) = \int_{-\infty}^{\infty} \hat{f}_n(\xi) e^{2\pi i x\xi}\,d\xi \overset{DCT}{\Rightarrow} f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi}\,d\xi$

<u>Laplace Transform</u>: $f:[0,\infty)\to\mathbb{C}$

$(\mathcal{L}f)(\lambda) = \int_0^{\infty} f(x) e^{-\lambda x}\,dx$

$\mathrm{Re}(\lambda)>0$, f is holo. for nice f (ie, cts cpt support, bdd, ...)

Similarly to Fourier transform, $\mathcal{L}$ can be inverted via complex analysis. (the only way to invert $\mathcal{L}$ is via complex analysis).

<u>Poisson Summation</u>

<u>Thm</u>: If $f\in\mathcal{F}$, then $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$.

<u>Pf</u>: Suppose $f\in\mathcal{F}_a$. Choose b w/ $0<b<a$. The fcn $\frac{1}{e^{2\pi i z}-1}$ has poles when $z\in\mathbb{Z}$. This is a simple pole: $\frac{d}{dz}(e^{2\pi i z}-1)\Big|_{z=n} = 2\pi i e^{2\pi i z}\Big|_{z=n} = 2\pi i \ne 0$

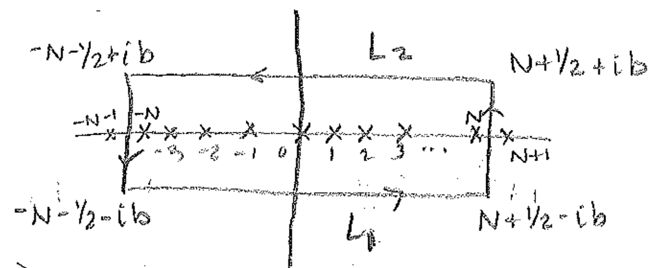
$\Rightarrow e^{2\pi i z}-1$ has a simple zero

$\Rightarrow \frac{1}{e^{2\pi i z}-1}$ has simple pole w/ residue $1/2\pi i$.

So $\frac{f(z)}{e^{2\pi i z}-1}$ has simple poles at $z \in \mathbb{Z}$ w/ res $\frac{f(n)}{2\pi i}$

(If $f(n)=0$, it's a pole w/ res $=0$, ie, not a pole)

Consider γ_N:



$$\int_{\gamma_N} \frac{f(z)}{e^{2\pi i z}-1}\,dz$$

$$2\pi i \sum_{|n|\le N} \frac{f(n)}{2\pi i} = \sum_{|n|\le N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi i z}-1}\,dz$$

On vertical strips, $\left|\frac{1}{e^{2\pi i z}-1}\right| \le C$, which is ind. of N

(b/c it's periodic ie: $e^{2\pi i z} = e^{2\pi i (z+1)}$)

Also, $|f(x+iy)| \le \frac{A}{x^2+1}$, so as $n \to \infty$, the $\int$'s over the vertical edges $\to 0$.

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi i z}-1}\,dz - \int_{L_2} \frac{f(z)}{e^{2\pi i z}-1}\,dz$$

$L_1 = \mathbb{R} - ib$, $L_2 = \mathbb{R} + ib$ going from left to right

$|e^{2\pi i z}|>1$ $\quad$ $|e^{2\pi i z}|<1$

L_1: If $|w|>1$, $\frac{1}{w-1} = \frac{1}{w}\cdot\frac{1}{1-1/w} = \frac{1}{w}\cdot\sum_{n=0}^{\infty} w^{-n}$

So on L_1, where $|e^{2\pi i z}|>1$ (b/c $\operatorname{Im}(z)<0$),

$$\frac{1}{e^{2\pi i z}-1} = e^{-2\pi i z}\sum_{n=0}^{\infty} e^{-2\pi i n z}$$

L_2: If $|w|<1$, $\frac{1}{w-1} = -\frac{1}{1-w} = -\sum_{n=0}^{\infty} w^n$

$$\frac{1}{e^{2\pi i z}-1} = -\sum_{n=0}^{\infty} e^{2\pi i n z}$$

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} f(z)\left(e^{-2\pi i z}\sum_{n=0}^{\infty} e^{-2\pi i n z}\right)dz + \int_{L_2} f(z)\sum_{n=0}^{\infty} e^{2\pi i n z}\,dz$$

$$= \sum_{n=0}^{\infty} \int_{L_1} f(z) e^{-2\pi i (n+1) z}\,dz + \sum_{n=0}^{\infty} \int_{L_2} f(z) e^{2\pi i n z}\,dz$$

want to move L_1 & L_2 back to $\mathbb{R}$: can shift contour up/down if can bound edges. OR true $\forall b$, so can take limit as $b \to 0$.

$$= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-2\pi i(n+1)x}\, dx + \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f(x) e^{2\pi i n x}\, dx$$
$\uparrow \hat{f}$ at $n < -1$ $\qquad \uparrow \hat{f}$ at $n \geq 0$

$$= \sum_{n \in \mathbb{Z}} \hat{f}(n) \qquad \square$$

$\boxed{\text{Paley-Weiner Thm}}$ Assume we know Fourier inversion for $|f(x)| \leq \frac{A}{1+x^2}$, $|\hat{f}(\xi)| \leq \frac{A}{1+\xi^2}$, so $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi}\, d\xi$

f supp. in $[-M, M]$. $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx = \int_{-M}^{M} f(x) e^{-2\pi i x \xi}\, dx$

can use Fubini to show $\hat{f}$ entire in ξ.
Ie, Fourier transform of cptly supp. fcn is entire.
Given an entire fcn, how can you tell if it's the F. transf. of a cptly supp. fcn?

Thm: Suppose $\hat{f}$ satisfies $|\hat{f}(\xi)| \leq A e^{-2\pi a|\xi|}$ for some $A > 0$, $a > 0$.
[If $f \in \mathfrak{F}_a$, then $\hat{f}$ satisfies this $\forall\, b < a$]
Then $f(x)$ is the restriction to $\mathbb{R}$ of a fcn $f(z)$ holo. in the strip $S_b = \{z \in \mathbb{C} : |\text{Im}(z)| < b\}$ $\forall\, 0 < b < a$.

4/12 * how smooth $\hat{f}$ is depends on how fast f decays at ∞
- If f decays faster than any poly, then $\hat{f}$ is smooth.
* If $\hat{f}$ decays exp'lly. (f smooth), then f real analytic & can extend to holo. on strip

Pf: We know $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x}\, d\xi$. Extend this to $f(z) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi z}\, d\xi$.
If $z \in S_b$, then $\qquad$ Re part involves $\text{Im}(z) <$ bdd
$$\int_{-\infty}^{\infty} |\hat{f}(\xi)| |e^{2\pi i \xi z}|\, d\xi \leq A \int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{2\pi \xi b}\, d\xi < \infty$$
($b < a$, so this finite)
We have defined $f(z)$ for $z \in S_b$. If γ is a closed curve in S_B,

$\int_\gamma f(z)\,dz \overset{\text{Fubini}}{=} \int_{-\infty}^{\infty} \hat{f}(\xi) \int_\gamma e^{2\pi i \xi z}\,dz\,d\xi = 0$, so by Morera's Thm f holomorphic on S_b.

Cor: If $|\hat{f}(\xi)| \le C e^{-2\pi a|\xi|}$ for some $a>0$, & f vanishes in a non-empty open interval. Then $f \equiv 0$.

Pf: f holo. on strip containing $\mathbb{R}$, & every open interval has a limit pt $\Rightarrow f \equiv 0$.

Cor: If f & $\hat{f}$ both have cpt support, then $f \equiv 0$.

Pf: $\hat{f}$ cpt supp $\Rightarrow \hat{f}$ falls off exponentially (can pick any a)

f cpt supp $\Rightarrow f$ vanishes on a non-empty open interval

$\Rightarrow$ use prev. cor.

Called the Heisenberg Uncertainty Principle

* If $\hat{f}$ cpt supp $\Rightarrow$ can choose any a & that determines the size of strip in which f holo $\Rightarrow f$ actually entire

Thm: Suppose f is cts & $|f(x)| \le \frac{A}{1+x^2}$. TFAE:

(1) f has an entire extension w/ $|f(z)| \le A e^{2\pi M|z|}$.

(2) $\hat{f}$ is supported on $[-M, M]$

Pf: (ii) $\Rightarrow$ (i): Suppose $\hat{f}$ is supp. in $[-M,M]$. Then $|f(x)| \le \frac{A}{1+x^2}$ & $|\hat{f}(\xi)| \le \frac{A}{1+\xi^2}$ (can use Fourier inversion)

& $f(x) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i \xi x}\,d\xi$ ($\hat{f}$ bdd by L^1 norm of f)

Define $g(z) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i \xi z}\,d\xi$ ok (z fixed, makes sense for every z b/c $\int_{-M}^{M}$ not $\int_{-\infty}^{\infty}$)

By Morera's thm, g is entire & $g(z) = f(z)$ when $z \in \mathbb{R}$. If $z = x+iy$

$|g(z)| \le \int_{-M}^{M} |\hat{f}(\xi)|\, e^{-2\pi \xi y}\,d\xi \le A e^{2\pi M|y|} \le A e^{2\pi M|z|}$ ✓

bound of $\hat{f}$, $\int e^{-2\pi\xi y}$

$(i) \Rightarrow (ii)$:

<u>Step 1</u>: We assume f is entire &

$$|f(x+iy)| \le A' \frac{e^{2\pi M|y|}}{1+x^2} \qquad (f \text{ is } L^1 \text{ in } x \text{ for each fixed } y)$$

Goal: Show $\hat{f}(\xi) = 0$ if $|\xi| > M$

If $\xi > M$,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x}\, dx$$ (to replace x w/ z, if $\xi > M$, $e^{\ldots}$ ok for $\mathrm{Im}(z) <$ & better for $\mathrm{Im}(z) \ll 0$

Shift contour down:

$$= \int_{-\infty}^{\infty} f(x-iy) e^{-2\pi i \xi (x-iy)}\, dx$$ (use contour

$\leftarrow$ goes to

since $|f(x+iy)| \le A'(y) \cdot \frac{1}{1+x^2}$

const. depends on y

$$|\hat{f}(\xi)| \le A' \int_{-\infty}^{\infty} \frac{e^{2\pi M y - 2\pi \xi y}}{1+x^2}\, dx$$ ($y > 0$) anything we want

$$\le A'' e^{-2\pi y(\xi - M)} \xrightarrow{y \to \infty} 0$$ (>0 b/c $\xi > M$)

If $\xi < -M$, just shift contour up & make same argument

$\Rightarrow \hat{f}(\xi) = 0$.

<u>Step 2</u>: Assume $|f(x+iy)| \le A e^{2\pi M|y|}$. Suppose $\xi > M$. We wish to show $\hat{f}(\xi) = 0$.

For $\varepsilon > 0$ consider

$$f_\varepsilon(z) := \frac{f(z)}{(1+i\varepsilon z)^2}$$. If $\mathrm{Im}(z) < 0$, denom > 1.

$\left|\frac{1}{(1+i\varepsilon z)^2}\right| \le 1$ in the lower half plane (including $\mathbb{R}$).

<u>Note</u>: $|\hat{f}_\varepsilon(\xi) - \hat{f}(\xi)| \le \int_{-\infty}^{\infty} \underbrace{|f(x)|}_{L^1} \cdot \left|\frac{1}{(1+i\varepsilon x)^2} - 1\right| dx$ (the $e^{\ldots}$ has modulus 1; ≤ 1)

by DCT: $\longrightarrow 0$ as $\varepsilon \to 0$. For $y \le 0$,

$$|f_\varepsilon(x+iy)| \le A \frac{e^{2\pi M|y|}}{|1+i\varepsilon z|^2} \le A \frac{e^{2\pi M|y|}}{|1+i\varepsilon x|^2} \le A_\varepsilon \frac{e^{2\pi M|y|}}{1+x^2}$$

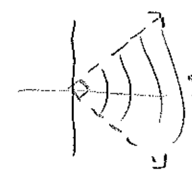
$\Rightarrow \hat{f}_\varepsilon(\xi) = 0$ for $\xi > M$ (by step 1)

$\varepsilon \to 0$ & get $\hat{f}(\xi) = 0$. (by note),

For $\xi < -M$, use $\frac{f(z)}{(1-i\varepsilon z)^2}$ in the upper half plane.

Step 3: If $|f(x)| \le 1 \ \forall x \in \mathbb{R}$ and $|f(z)| \le e^{2\pi M|z|} \ \forall z \in \mathbb{C}$.
(ie mult f by constant so all constants in bounds become 1)
Then we want to show $|f(x+iy)| \le e^{2\pi M|y|}$

4/15 (Phragmen & Lindelöf Type) Thm: Suppose F is a holo. fcn in the sector $S = \{z \mid -\pi/4 < \operatorname{Arg} z < \pi/4\}$ & F is cts on $\overline{S}$. Assume $|F(z)| \le 1$ on ∂S, & $\exists\, c, C > 0$ s.t. $|F(z)| \le Ce^{c|z|} \ \forall z \in S$. Then $|F(z)| \le 1 \ \forall z \in S$.


S

(could rotate sector by composing w/ rotation & thm will apply; can have any power of $z < 2$)
(can take diff ∡'s, & that changes the hypothesis of the growth — diff powers of $|z|$).

Ex: $F(z) = e^{z^2}$. If $\operatorname{Arg} z = \pm\pi/4$, $z^2 \in i\mathbb{R}$, so on ∂S, $|F(z)| = 1$. But on $\mathbb{R}$, $F(x) = e^{x^2}$, which is big.

Pf: Let $F_\varepsilon(z) = F(z)e^{-\varepsilon z^{3/2}}$. Here $z^{3/2}$ is defined via the principal branch of the logarithm:

($3/2$ could be any # btwn 1 & 2) ↑ ✓ ↑ won't work

If $z = re^{i\theta}$, $(-\pi < \theta < \pi)$, then $z^{3/2} = r^{3/2}e^{i3\theta/2}$; defined on S. F_ε is holo. on S. $z^{3/2} \to 0$ as $z \to 0$ in S, so F_ε is cts on $\overline{S}$ (even though not holo. on any nbhd of 0)

$|e^{-\varepsilon z^{3/2}}| = e^{-\varepsilon r^{3/2}\cos(3\theta/2)}$ ← $\operatorname{Re}(z^{3/2})$. On S, $-\pi/4 < \theta < \pi/4$, so

$\underset{\cos = 0}{-\pi/2} < -3\pi/8 < 3\theta/2 < 3\pi/8 < \underset{\cos = 0}{\pi/2}$ $\Rightarrow \cos(3\theta/2)$ stays away from 0 on $S \Rightarrow$ On S, $|\cos(3\theta/2)| \ge b > 0$.

$|F_\varepsilon(z)| \le Ce^{c|z| - \varepsilon|z|^{3/2}\cos(3\theta/2)} \le Ce^{c|z| - \varepsilon b|z|^{3/2}} \to 0$ as $z \to \infty$

(dominates)

We conclude $|F_\varepsilon(z)| \to 0$ as $z \to \infty$ in $\overline{S}$.
Thus F_ε is bdd.

Claim: $|F_\varepsilon(z)| \le 1 \ \forall z \in \overline{S}$.

Define $M = \sup_{z \in \overline{S}} |F_\varepsilon(z)|$. We assume $M > 0$ (else $F_\varepsilon \equiv 0 \Rightarrow F \equiv 0 \Rightarrow$ trivial)

Let $\{w_j\} \subseteq S$ be a sequence of pts s.t. $|F_\varepsilon(w_j)| \to M$
The w_j's are bdd, as otherwise M would be zero & $F \equiv 0$. The w_j's have a limit pt $w \in \overline{S}$ w/ $|F_\varepsilon(w)| = M$ (b/c F_ε cts). (apply max-mod principle on a bdd set $\subseteq S$ $\triangleleft$ b/c F_ε small for z large, $\Rightarrow w$ can't be on $\triangleleft^c$ b/c F_ε small, so must be $\triangleleft$)
So $w \in \partial S$, But $|F_\varepsilon(z)| \leq 1$ on ∂S by assumption.
We conclude $M \leq 1$, i.e. $|F_\varepsilon(z)| \leq 1$ $\forall z \in S$.
$F_\varepsilon(z) = F(z) e^{-\varepsilon z^{3/2}} \xrightarrow{\varepsilon \to 0} F(z)$. We conclude $|F(z)| \leq 1$ on S.

<u>Payley-Weiner Thm</u>: Suppose $f: \mathbb{R} \to \mathbb{C}$ cts w/ $|f(x)| \leq \frac{A}{1+x^2}$.
TFAE:
(i) f has an entire ext'n w/ $|f(z)| \leq A e^{2\pi M |z|}$
(ii) $\hat{f}$ is supp. in $[-M, M]$.
(ie, fcns w/ an entire ext'n of exponential type are those whose Fourier transf. is supp. in $[-M, M]$.
(ii) $\Rightarrow$ (i): easy
(i) $\Rightarrow$ (ii): 3 steps
① We assume $|f(x+iy)| \leq A' \frac{e^{2\pi M|y|}}{1+x^2}$, prove (ii)
② We assume $|f(x+iy)| \leq A e^{2\pi M|y|}$, prove (ii)
③ We assume $|f(z)| \leq A e^{2\pi M|z|}$ & prove the assumption in ② holds. We suppose f is entire, $|f(x)| \leq 1$ for $x \in \mathbb{R}$, & $|f(z)| \leq e^{2\pi M|z|}$.
<u>Claim</u>: $|f(z)| \leq e^{2\pi M|y|}$
Apply Phragmen-Lindelöf to $Q = \{x+iy \mid x>0, y>0\}$
(same proof will work in each of 4 quadrants)
Consider $F(z) = f(z) e^{2\pi i M z}$. For $x>0$, $|F(x)| \leq 1$ by our assumption. Also, for $y>0$,
$|F(iy)| \leq |f(iy)| e^{-2\pi M y} \leq e^{2\pi M y - 2\pi M y} = e^0 = 1$.
Finally, $|F(z)| \leq e^{2\pi M|z|} e^{2\pi M|z|} \leq e^{\overbrace{c|z|}^{4\pi M}}$.

$\overset{P\text{-}L}{\Longrightarrow}$ $|F(z)| \le 1$ on Q.

$1 \ge |F(z)| = |f(z)||e^{2\pi i M z}|$

$= |f(z)|\, e^{-2\pi M y}$, $y > 0$ b/c in Q

We see $|f(z)| \le e^{2\pi M |y|}$ for $z \in Q$.

Same for other 3 quadrants.

This completes the proof of the P-W Thm $\square$

Thm: Suppose $|f(x)| \le \frac{A}{1+x^2}$ & $|\hat{f}(\xi)| \le \frac{A}{1+\xi^2}$, $f: \mathbb{R} \to \mathbb{C}$.

TFAE:

(i) $\hat{f}(\xi) = 0 \quad \forall\, \xi < 0$.

(ii) f can be extended to a cts & bdd fcn in the closed upper 1/2 plane $\{z = x+iy \mid y \ge 0\}$ w/ f holo. in the open upper 1/2 plane.

Pf: (i) $\Rightarrow$ (ii): $f(x) = \int_0^\infty \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi$.

For $y > 0$, define

$$f(z) = \int_0^\infty \hat{f}(\xi) \underbrace{e^{2\pi i z \xi}}_{\text{in modulus } = e^{-2\pi y \xi}} d\xi \qquad \text{but } \xi > 0 \text{, so no problem,}$$

& f holo. by Morera & Fubini for $y > 0$ & by DCT $f(z)$ is cts for $y \ge 0$ (take a limit)

(ii) $\Rightarrow$ (i): Set $f_{\varepsilon,\delta}(z) = \frac{f(z+i\delta)}{(1-i\varepsilon z)^2}$, $\varepsilon, \delta > 0$.

$f_{\varepsilon,\delta}(z)$ is holo. in a region containing the closed UHP.

Claim: $\hat{f}_{\varepsilon,\delta}(\xi) = 0$ for $\xi < 0$.

$\hat{f}_{\varepsilon,\delta}(\xi) = \int_{-\infty}^{\infty} \frac{f(x+i\delta)}{(1-i\varepsilon x)^2} e^{-2\pi i x \xi}\, dx$. Can shift contour up:

$$= \int_{-\infty}^{\infty} \frac{f(x+i\delta+iy)}{(1-i\varepsilon(x+iy))^2} e^{-2\pi i x \xi} \underbrace{e^{2\pi y \xi}}_{\to 0 \text{ as } y \to \infty \text{ b/c } \xi < 0} dx$$

$\therefore$ for $\xi < 0$, $\hat{f}_{\varepsilon,\delta}(\xi) = 0$.

Take $\delta \to 0$ shows $\hat{f}_{\varepsilon,0}(\xi) = 0$ for $\xi < 0$

Take $\varepsilon \to 0$ shows $\hat{f}_{0,0}(\xi) = 0 \Rightarrow \hat{f}(\xi) = 0$.

4/17 <u>Mean Value Property</u>

Harmonic $\Leftrightarrow$ $\Delta u = 0$ $\Leftrightarrow$ $u = \mathrm{Re} f$, f is holo.

① If f is holo. on a disk $\{|z - z_0| \le r\}$, then

$f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z - z_0}\,dz$. use $\gamma(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$$

$$\mathrm{Re}(f(z_0)) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}(f(z_0 + re^{i\theta}))\,d\theta.$$

If $u = \mathrm{Re} f$,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta$$

↑ value at center of ⊙ = ave value of u on the ⊙.

<u>Thm</u>: (Jensen's formula). $D_R = \{z \mid |z| < r\}$, $C_R = \partial D_R$

Let Ω be an open set $\overline{D_R} \subseteq \Omega$ & suppose f is holo in Ω, $f(0) \ne 0$ & f does not vanish on C_R. Let $z_1, \dots, z_N$ be the zeros of f in D_R. Then

$$\log|f(0)| = \sum_{i=1}^{N} \log\left(\frac{|z_i|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta \quad (\dagger)$$

(ie, can almost pull log into $\int$, but get error (the sum)).

Suppose f_1, f_2 satisfy the hypotheses of the thm & satisfy $(\dagger)$. Consider $g(z) = f_1(z) f_2(z)$. The zeros of g are the union of the zeros of f_1 & f_2. $\log|g(z)| = \log|f_1(z)| + \log|f_2(z)|$. So $(\dagger)$ for g follows from $(\dagger)$ for f_1 & f_2 (ie, the sum)

<u>Pf</u>: $g(z) = \frac{f(z)}{(z - z_1)\cdots(z - z_N)}$ is holo. in Ω & never zero in D_R. $f(z) = (z - z_1)\cdots(z - z_N) g(z)$. We'll show $(\dagger)$ holds for g & for $(z - z_k)$. (Then have $(\dagger)$ for f by additivity of conclusion).

Pf of (†) for g: WTS $\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\,d\theta$

If $\log|g(z)|$ harmonic, this follows from earlier result.

g is never zero in $\overline{D_R}$, so g is holo & $\neq 0$ on a nbhd of $\overline{D_R}$. On a nbhd of $\overline{D_R}$, $g(z) = e^{h(z)}$, $h(z)$ holo. Call $h = \log g$.

(real part of h ! determined, & im part can vary by multiples of $2\pi i$).

$|g(z)| = |e^{h(z)}| = e^{Re(h(z))} \Rightarrow$

$\log|g(z)| = Re(h(z))$. We conclude $\log|g(z)|$ is harmonic b/c real part of holo. fcn. By the MVP, the conclusion holds.

Pf of (†) for $f(z) = (z-w)$, $w \in D_R$:

ie, WTS $\log|w| = \log\frac{|w|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - w|\,d\theta$

$\log\frac{|w|}{R} = \log|w| - \log R$ and $\log|Re^{i\theta} - w| =$

$\log R + \log|e^{i\theta} - \frac{w}{R}|$

$\Rightarrow$ WTS $\int_0^{2\pi} \log|e^{i\theta} - a|\,d\theta = 0$, $a = \frac{w}{R}$ & $|a| < 1$.

i.e. WTS $\int_0^{2\pi} \log|e^{-i\theta} - a|\,d\theta = 0$

$\qquad\qquad \shortparallel \qquad \downarrow |e^{-i\theta}| = 1$

Goal: $\int_0^{2\pi} \log|1 - ae^{i\theta}|\,d\theta$

Consider $F(z) = 1 - az$, $F(z)$ is nonzero on D_1

We know $\underbrace{\log|F(0)|}_{\log 1 = 0} = \frac{1}{2\pi}\int_0^{2\pi} \log|F(e^{i\theta})|\,d\theta$ ✓

□

Suppose f is holo. in D_R & $0 < r < R$. Let $n(r) = \#$ of zeros of f in D_r. Notice $n(r)$ is non-decreasing in r.

Suppose $f(0) \neq 0$ & f does not vanish on C_R. Then

$$\int_0^R n(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)| \quad (n(r) = 0 \text{ for } r \text{ small})$$

<u>Lemma</u>: If $z_1, \dots, z_N$ are the zeros of f in D_R, then

$$\int_0^R n(r)\,\frac{dr}{r} = \sum_{k=1}^N \log\left|\frac{R}{z_k}\right|.$$

<u>Pf</u>: $$\sum_{k=1}^N \log\left|\frac{R}{z_k}\right| = \sum_{k=1}^N \overbrace{\int_{|z_k|}^R \frac{dr}{r}}^{=\log\left|\frac{R}{z_k}\right|}$$

$$\eta_k(r) = \begin{cases} 1 & \text{if } r > |z_k| \\ 0 & \text{if } r \le |z_k| \end{cases}$$

$$= \sum_{k=1}^N \int_0^R \eta_k(r)\,\frac{dr}{r} = \int_0^R \sum_{k=1}^N \eta_k(r)\,\frac{dr}{r} = \int_0^R n(r)\,\frac{dr}{r} \qquad \square$$

<u>Def</u>: An entire fcn has growth $\le \rho$ if $|f(z)| \le A e^{B|z|^\rho}$.
The <u>order of growth</u> of f is the infimum of all such ρ.

ex: $(1+z^2+z^3)e^{1+z+z^2}$ has order of growth 2.

<u>Thm</u>: If f is an entire fcn that has an order of growth $\le \rho$, then:

(i) $n(r) \le C r^\rho$, $\forall r$ sufficiently large

(ii) If $z_1, z_2, \dots$ are the zeros of f, which are non-zero, then $\forall s > \rho$, $\sum_{k=1}^\infty \frac{1}{|z_k|^s} < \infty$. (ie, zeros go to ∞ quickly so sum conv.)

<u>Pf</u>: Set $g(z) = \frac{f(z)}{z^m}$ where m is the order of the zero of f at 0. g is entire, & for $|z| > 1$, $|g(z)| \le |f(z)|$, so g has same order of growth as f, ie $\le \rho$.
$n_g(r) = n_f(r) - m$, (ii) doesn't mention zeros at 0.
Thus it suffices to prove thm for f non-zero at 0, so assume $f(0) \ne 0$. Then

$$\int_0^R n(x)\,\frac{dx}{x} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|$$

Let $R = 2r$. b/c $\ge$ non-decr.

$$\int_r^{2r} n(x)\,\frac{dx}{x} \le \int_0^R n(x)\,\frac{dx}{x} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|$$

smallest value of $x \mapsto n(x)$ non-decr.

So $$\int_r^{2r} n(x)\,\frac{dx}{x} \ge n(r)\int_r^{2r} \frac{dx}{x} = n(r)\big(\log(2r) - \log(r)\big) = n(r)\log 2.$$

WTS $\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta \le r^\rho$.

$$\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta \le \int_0^{2\pi} \log|Ae^{Br^\rho}|\,d\theta$$
$$\le CR^\rho = C(2r)^\rho = C'r^\rho.$$
$\Rightarrow n(r) \le C'' r^\rho.$ (i) ✓

4/19 A fcn has an order of growth $\le \rho$ if $|f(z)| \le Ae^{B|z|^\rho}$.
The order of growth of f is the infimum of such ρ.

<u>Thm</u>: If f is an entire fcn that has an order of growth $\le \rho$, then
(i) $n(r) \le C r^\rho$, for some $C>0$, $r \ge 1$.
(ii) If $z_1, z_2, \dots$ are the nonzero zeros of f, then
$\forall s > \rho$, $\sum_{k=1}^\infty \frac{1}{|z_k|^s} < \infty$.

<u>Pf</u>: (i) done

double counting

(ii) $$\sum_{|z_k|\ge 1} |z_k|^{-s} = \sum_{j=0}^\infty \Big(\sum_{2^j \le |z_k| < 2^{j+1}} |z_k|^{-s} \Big) \le \sum_{j=0}^\infty 2^{-js} n(2^{j+1})$$
$$\le C \sum_{j=0}^\infty 2^{-js} 2^{(j+1)\rho} < \infty \quad \text{(geom. sum \& } s > \rho)$$

<u>Ex</u> ① $f(z) = \sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$
$|f(z)| \le e^{\pi z} \Rightarrow$ the order of growth is 1 :
$z = ix$, $x \in \mathbb{R}$, $\sin(\pi z) = \frac{e^{-\pi x} - e^{\pi x}}{2i}$ = order of growth = 1
(when x large, $= e^{\pi z}$)
$\sin(\pi z)$ has a zero of order 1 at $z = n$.
$\sum_{n \ne 0} \frac{1}{n^s} < \infty \Leftrightarrow s > 1$.

② $\cos(z^{1/2}) := \sum_{n=0}^\infty (-1)^n \frac{z^n}{(2n)!}$ ← replace z^2 w/ z in power series
→ conv. abs. $\forall z$ b/c ! beats exponential
$|\cos(z^{1/2})| \le \sum_{n=0}^\infty \frac{|z|^n}{(2n)!} = e^{|z|^{1/2}}$ so an order of growth $\le 1/2$

$z_n = ((n+1/2)\pi)^2$ are zeros for $\cos(z^{1/2})$
$\sum_n \frac{1}{|z_n|^s} < \infty$ iff $s > 1/2$ $\Rightarrow$ the order of growth is $1/2$.
↑ more or less n^2

Recall: Weierstrass factorization thm: For each $k \geq 0$, $E_0(z) = 1-z$, $E_k(z) = (1-z)e^{z + z^2/2 + \cdots + z^k/k}$.
Given any sequence $\{a_n\} \subseteq \mathbb{C}$ w/ $|a_n| \to \infty$, $\exists$ an entire fcn f that vanishes precisely at the a_n's.
Any other such entire fcn is $e^{g(z)} \cdot f(z)$ where g is entire. m = order of zero at $z=0$

$$f(z) = z^m \prod_n E_n(z/a_n)$$

n is the degree of E_n.

When f is a fcn of finite order,
(i) the degree of the canonical factors can be chosen independent of n.
(ii) g is a polynomial.

Thm: Suppose f is entire & has growth order ρ_0. Let k be the integer s.t. $k \leq \rho_0 < k+1$ ($k = \lfloor \rho_0 \rfloor$), if $a_1, a_2, \ldots$ are the nonzero zeros of f, then

$$f(z) = e^{P(z)} \cdot z^m \cdot \prod_{n=1}^{\infty} E_k(z/a_n)$$

m = deg. of zero at 0
P is a polynomial of deg $\leq k$.

Lemma: The canonical products satisfy

$$|E_k(z)| \geq e^{-c|z|^{k+1}} \text{ for } |z| \leq 1/2,$$

$$|E_k(z)| \geq |1-z|\, e^{-c'|z|^k} \text{ for } |z| > 1/2$$

Pf: If $|z| \leq 1/2$, $\log(1-z) = \sum_1 -\frac{z^n}{n}$

$$E_k(z) = (1-z)e^{z + z^2/2 + \cdots + z^k/k}$$
$$= e^{\sum -z^n/n + z + z^2/2 + \cdots + z^k/k}$$
$$= e^{-\sum_{n=k+1}^{\infty} z^n/n}$$
$$= e^{w}$$

$|e^z| \leq e^{|z|}$, $|e^z| \geq e^{-|z|}$

$$|w| = \left|-\sum_{n=k+1}^{\infty} z^n/n\right| \leq c|z|^{k+1}$$

$\Rightarrow |E_k(z)| \geq e^{-|w|} \geq e^{-c|z|^{k+1}}$

• geom sum bdd by const. times largest term

If $|z| > 1/2$,

$$|E_k(z)| = |1-z|\,\left|e^{z+z^2/2+\cdots+z^k/k}\right| \quad \text{bdd by const. times } |z|^k$$

$$\geq |1-z|\, e^{-|z+z^2/2+\cdots+z^k/k|}$$

$$\geq |1-z| e^{-c'|z|^k} \qquad \square$$

__Lemma__: For any s with $\rho_0 < s < k+1$,

$\left|\prod_{n=1}^{\infty} E_k(z/a_n)\right| \geq e^{-c|z|^s}$, except possibly when z

belongs to a disk centered at a_n w/ radius $|a_n|^{-k-1}$.

(ie, if z doesn't get too close to any zero)

__Pf__: $\prod_{n=1}^{\infty} E_k(z/a_n) = \underbrace{\left[\prod_{|a_n|<2|z|} E_k(z/a_n)\right]}_{①} \underbrace{\left[\prod_{|a_n|>2|z|} E_k(z/a_n)\right]}_{②}$

Want to prove the lower bd for each factor separately.

② $\left|\prod_{|a_n|>2|z|} E_k(z/a_n)\right| = \prod_{|a_n|>2|z|} |E_k(z/a_n)| \geq \prod_{|a_n|>2|z|} e^{-c|z/a_n|^{k+1}}$ from prev. lemma.

$$\geq e^{-c|z|^{k+1} \sum_{|a_n|>2|z|} \frac{1}{|a_n|^{k+1}}} \quad \text{ie, } |z/a_n| < 1/2 \qquad (\prod e = e^{\Sigma}) \qquad s < k+1$$

$$|a_n|^{-k-1} = \underbrace{|a_n|^{-s}}_{\text{summable}} |a_n|^{s-k-1} \leq c|a_n|^{-s}|z|^{s-k-1}$$

$\uparrow$ $|a_n| > 2|z|$, so replace a_n w/ z b/c negative power $(s-k-1)$

$$\Rightarrow e^{-c|z|^{k+1}\sum \frac{1}{|a_n|^{k+1}}} \geq e^{-c'|z|^s \sum \frac{1}{|a_n|^s}} \quad \text{summable}$$

$$\geq e^{-c''|z|^s}$$

① $\left|\prod_{|a_n|\leq 2|z|} E_k(z/a_n)\right| = \prod_{|a_n|\leq 2|z|} |E_k(z/a_n)| \geq \underbrace{\prod_{|a_n|\leq 2|z|} \left|1-\frac{z}{a_n}\right|}_{(a)} \underbrace{\prod_{|a_n|\leq 2|z|} e^{-c'|\frac{z}{a_n}|^k}}_{(b)}$ $\leftarrow$ from prev. lemma

do (a), (b) separately.

4/22 $E_0(z) = 1-z$, $E_k(z) = (1-z)e^{z+z^2/2+\cdots+z^k/k}$

Thm: Suppose f is entire & has growth order ρ_0. Let k be the int. s.t. $k \le \rho_0 < k+1$. If $a_1, a_2, \ldots$ are the nonzero zeros of f, then

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k(z/a_n)$$ where P is a polynomial of deg. $\le k$

(e^{e^z} doesn't have a finite order of growth)

(Weierstrass: · P was just an entire fcn
· used E_n, not a fixed E_k)

New info: $\sum_i a_n^{-s} < \infty$ $\forall s > \rho_0$ (proven already) (†)
- this is all we'll use. - true $\forall$ fcns of finite growth

$E(z) = \prod_{n=1}^{\infty} E_k(z/a_n)$ converges like before & has zeros precisely at the a_n (using †).

$\dfrac{f(z)}{z^m E(z)} = e^{P(z)}$ (b/c quotient is nonzero).

Q: Why is P a polynomial?
- have upper bd for f, want lower bd for $E(z)$ (i.e. $E(z)$ not too small away from a_n's) s.t. P must be a poly.

Lemma: The canonical products satisfy
$|E_k(z)| \geq e^{-c|z|^{k+1}}$ if $|z| \le 1/2$
$|E_k(z)| \geq |1-z| e^{-c'|z|^k}$ if $|z| > 1/2$

Lemma: For any s w/ $\rho_0 < s < k+1$, we have
$\left|\prod_{n=1}^{\infty} E_k(z/a_n)\right| \geq e^{-c|z|^s}$ except possibly when z belongs to the union of disks centered at a_n w/ radius $|a_n|^{-k-1}$

Pf: Write $\prod_{n=1}^{\infty} E_k(z/a_n) = \Big[\underbrace{\prod_{|a_n|\le 2|z|} E_k(z/a_n)}_{①}\Big]\cdot\Big[\underbrace{\prod_{|a_n|>2|z|} E_k(z/a_n)}_{②}\Big]$

Last time, showed $|②| \ge e^{-C|z|^s}$

For ①,

$$\Big|\prod_{|a_n|\le 2|z|} E_k(z/a_n)\Big| \ge \underbrace{\prod_{|a_n|\le 2|z|}\Big|1-\frac{z}{a_n}\Big|}_{ⓐ} \cdot \underbrace{\prod_{|a_n|\le 2|z|} e^{-c|z/a_n|^k}}_{ⓑ}$$

For ⓑ: $\prod_{|a_n|\le 2|z|} e^{-c|z/a_n|^k} = e^{-c|z|^k \sum_{|a_n|\le 2|z|} |a_n|^{-k}}$

replace $|a_n|$ w/ $2|z|$

$$|a_n|^{-k} = |a_n|^{-s}|a_n|^{\overbrace{s-k}^{>0}} \le C|a_n|^{-s}|z|^{s-k}$$

$$e^{-c|z|^k \sum |a_n|^{-k}} \ge e^{-c'|z|^s \overbrace{\sum |a_n|^{-s}}^{<\infty}} \ge e^{-c''|z|^s} \checkmark$$

For ⓐ: Suppose $z \notin D_{|a_n|^{-k-1}}(a_n)$ $\forall n$, ie $|a_n - z| \ge |a_n|^{-k-1}$. Then

$$\prod_{|a_n|\le 2|z|} |1 - z/a_n| = \prod_{|a_n|\le 2|z|}\Big|\frac{a_n - z}{a_n}\Big| \ge \prod_{|a_n|\le 2|z|} |a_n|^{-k-1}|a_n|^{-1} = \prod_{|a_n|\le 2|z|} |a_n|^{-k-2}$$

want to bd from above b/c took out a ⊖

$$(k+2)\sum_{|a_n|\le 2|z|} \log|a_n| \le (k+2)\sum_{|a_n|\le 2|z|} \log 2|z| = (k+2)\log 2|z| \underbrace{n(2|z|)}_{\text{have bd}}$$

$$\le C|z|^s \log 2|z| \le C_{s,s'}|z|^{s'} \quad \forall s'>s$$

s was any $\# > \rho_0$ & $s'>s \Rightarrow$ shrink s & shrink s' to s, so this works $\forall s' > \rho_0$.

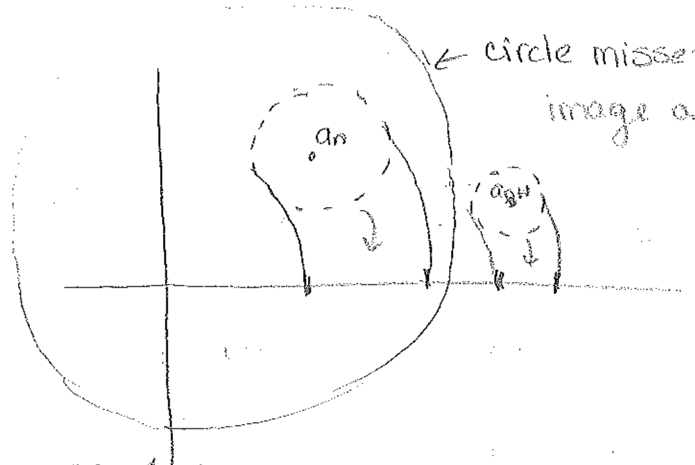
We get:

$$\prod_{|a_n|\le 2|z|} |1 - z/a_n| \ge e^{-c|z|^s} \quad \text{as desired} \checkmark$$

Cor: $\exists$ a sequence of radii $r_1, r_2, \ldots$ w/ $r_m \to \infty$ s.t. $\big|\prod_{n=1}^{\infty} E_k(z/a_n)\big| \ge e^{-c|z|^s}$ $\forall |z| = r_m$.

Pf: Since $\sum |a_n|^{-k-1} < \infty$, $\exists N$ w/ $\sum_{n=N}^{\infty} |a_n|^{-k-1} < 1/10$.

Claim: Given L sufficiently large, $\exists r$ w/ $L \le r \le L+1$ s.t. the circle of radius r does not intersect the disks of radius $|a_n|^{-k-1}$ centered at a_n.



← circle misses disks if it misses their image after rotation onto x-axis

Pf of Claim:

If there is no such r btwn L & $L+1$, the union of the intervals $I_n = [|a_n| - \frac{1}{|a_n|^{k+1}}, |a_n| + \frac{1}{|a_n|^{k+1}}]$ must cover $[L, L+1]$. By taking L so large that $|a_n| < L$ $\forall n \le N \Rightarrow 2\sum_{n=N}^{\infty} |a_n|^{-k-1} \geq 1$ (b/c covers $[L, L+1]$) & so $1/5 \geq 1$, ↯. $\square$

Pf of Hadamard's Thm: Let $E(z) = z^m \prod_{n=1}^{\infty} E_k(z/a_n)$. As in pf of Weierstrass factorization, E is entire & has zeros at a_n (use $\sum |a_n|^{-k-1} < \infty$). Look at the entire fcn, never zero

$$\frac{f(z)}{E(z)} = e^{g(z)}, \quad g \text{ is entire.}$$

Why is g a polynomial?

$$|e^{g(z)}| = e^{Re(g(z))}$$
$$= \left|\frac{f(z)}{E(z)}\right| \le \frac{c' e^{c|z|^s}}{|E(z)|} \quad \forall s > \rho_0 \quad \le c' e^{c''|z|^s} \quad \forall |z| = r_m \quad \forall s > \rho_0$$

(our lower bd on E_k true for E, as well, since $E = z^m E_k$, & $|z|^m$ large)

Lemma: Suppose g is entire & $u = Re(g)$. Suppose $u(z) \le C r_m^s$, $\forall |z| = r_m$ for some seq $r_m \to \infty$. Then u is a polynomial.

4/24 Review: $E_0(z) = 1-z$, $E_k(z) = (1-z)e^{z + z^2/2 + \cdots + z^k/k}$

Suppose f is entire & has growth order ρ_0. Let k be the int. s.t. $k \le \rho_0 < k+1$. If $a_1, a_2, \dots$ are the non-zero zeros of f, then

$f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} E_k(z/a_n)$ where m is the order of the zero at 0 & P is a poly. of deg $\le k$.

We already showed for $s > \rho_0$,

$|z^m \prod_{n=1}^{\infty} E_n(z/a_n)| \ge e^{-c|z|^s}$ when $|z| = r_m$ & r_m is a sequence $r_m \to \infty$.

$$e^{g(z)} = \underbrace{\frac{f(z)}{z^m \prod_{n=1}^{\infty} E_k(z/a_n)}}_{\text{never zero entire fcn.}}$$

WTS g a poly.

If $|z| = r_m$,

$$|e^{g(z)}| = e^{Re(g(z))} = \left| \frac{f(z)}{z^m \prod_{n=1}^{\infty} E_k(z/a_n)} \right| \le \frac{Ce^{c|z|^s}}{e^{-c'|z|^s}} = Ce^{c''|z|^s}$$

Goal: we'll be done if this implies g is a poly. of deg $\le s$ ie, of deg. $\le k$.

or: f real part of entire fcn grows like poly, whole thing grows like poly → only need it on circles tending to ∞.

Lemma: Suppose g is entire & $u = Re(g)$. Suppose u satisfies $u(z) \le Cr^s$ when $|z| = r$, for some sequence of r's tending to ∞. Then g is a poly of deg $\le s$.

Pf: Expand g as a power series so that $g(z) = \sum_{n=0}^{\infty} a_n z^n$. Goal: Show $a_n = 0$ for $n > s$.

$$a_n = \frac{g^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{|z|=r} \frac{g(z)}{z^{n+1}}\, dz \qquad z = re^{i\theta} \quad 0 \le \theta \le 2\pi$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(re^{i\theta})}{r^{n+1} e^{i(n+1)\theta}} \cdot \cancel{r} i e^{i\theta}\, d\theta$$

If $n \le -1$, $\int_{|z|=r} \frac{g(z)}{z^{n+1}}\, dz = 0$ by Cauchy

$$\Rightarrow \underbrace{\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) e^{-in\theta}\, d\theta}_{\uparrow} = \begin{cases} a_n r^n, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

(Fourier coeffs on a circle → only has ⊕, not ⊖ coeffs)

$n<0$:

$$0 = \frac{1}{2\pi} \int_0^{2\pi} \overline{g(re^{i\theta})}\, e^{-in\theta}\, d\theta \quad \text{(taking conjugates)}$$

⇒

If $n>0$, since $2u = g+\bar{g}$, we see

$a_n r^n = \frac{1}{\pi}\int_0^{2\pi} u(re^{i\theta}) e^{-in\theta}\, d\theta$

If $n=0$, $2\,\mathrm{Re}(a_0) = \frac{1}{\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta$

* holo. fcns have ⊕ F. coeffs, anti-holo. fcns have ⊖ F coeffs

Reminder: for $n \neq 0$ $\int_0^{2\pi} e^{in\theta}\, d\theta = 0$

$n>0$: $a_n = \frac{1}{\pi r^n}\int_0^{2\pi} [u(re^{i\theta}) - \underbrace{Cr^s}_{\geq 0}] e^{-in\theta}\, d\theta$

$|a_n| \leq \frac{1}{\pi r^n}\int_0^{2\pi} |u(re^{i\theta}) - Cr^s|\, d\theta$ $\quad Cr^s > u(re^{i\theta})$, so

$= \frac{1}{\pi r^n}\int_0^{2\pi} (Cr^s - u(re^{i\theta}))\, d\theta$ ← this integral we did previously

$= 2Cr^{s-n} - 2\,\mathrm{Re}(a_0) r^{-n}$

If $n>s$ $\longrightarrow 0$ as $r \to \infty$

So $a_n = 0$ $\forall n>s$, ie g is a poly of deg $\leq s$. □

Doubly-Periodic Functions

We're interested in mero. fcns on $\mathbb{C}$ which have 2 periods, ω_1 & $\omega_2 \in \mathbb{C}$.

$f(z+\omega_1) = f(z)$ $\quad \forall z$

$f(z+\omega_2) = f(z)$

What if $\frac{\omega_1}{\omega_2} \in \mathbb{R}$? (ie, ω_1, ω_2 are lin. dependent over $\mathbb{R}$)

(a) $\frac{\omega_1}{\omega_2} \in \mathbb{Q}$. Then can write this as being periodic wrt $\gcd(\omega_1, \omega_2)$, so it's secretly singularly periodic.

(b) $\frac{\omega_1}{\omega_2} \in \mathbb{R}\setminus\mathbb{Q}$. Then the fcn is constant

We want ω_1, ω_2 to be lin. ind. $/\mathbb{R}$.

$\tau = \frac{\omega_2}{\omega_1} \notin \mathbb{R}$.

τ & $\frac{1}{\tau}$: the imaginary parts have different signs.

Wlog, $\mathrm{Im}(\tau) > 0$.

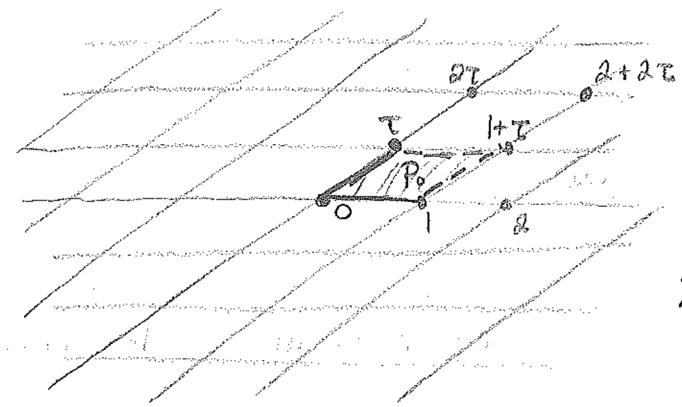
Define $F(z) = f(\omega_1 z)$

F has periods 1 & τ. Wlog, the pds of f are 1 & τ.

the only holo
cn here is
onst → max-
mod. principle
→ entire bdd)

<u>Note</u>: $f(z+n+m\tau) = f(z) \quad \forall n,m \in \mathbb{Z}$

Set $\Lambda = \{ n+m\tau \mid n,m \in \mathbb{Z} \}$. we say 1 & τ gen. Λ.



Λ is the lattice.

f is constant under translations by elts of Λ:
$\lambda \in \Lambda \Rightarrow f(z+\lambda) = f(z)$.

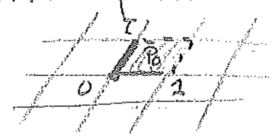
Let $P_0 = \{ z \in \mathbb{C} \mid z = a+b\tau,\ 0 \le a \le 1,\ 0 \le b \le 1 \}$. P_0 is called the fundamental ▱ for Λ.
f on P_0 determines f.

We say z & $w \in \mathbb{C}$ are congruent modulo Λ if
$z = w+n+m\tau$ for $n,m \in \mathbb{Z}$ (or equiv, if $z-w \in \Lambda$)
We write $z \sim w$. (Λ an additive subgp)
If $z \sim w$, then $f(z) = f(w)$.

<u>Claim</u>: Each $z \in \mathbb{C}$ is congruent to a unique pt in P_0.

<u>Pf</u>: Suppose $z = x+iy$. Write $z = a+b\tau$, $a,b \in \mathbb{R}$.
(ok b/c $\{1,\tau\}$ are a basis of $\mathbb{C}/\mathbb{R}$).
Choose $n,m \in \mathbb{Z}$ to be the greatest int $\le a,b$ (resp.) & let $w = z-n-m\tau$.
$z \sim w$, & $w = (a-n)+(b-m)\tau$ w/ $0 \le a-n < 1$ & $0 \le b-m < 1$, so $w \in P_0$.
For uniqueness, suppose $w-w' = (a-a')+(b-b')\tau \in \Lambda$ w/ $w,w' \in P_0$. $a-a', b-b' \in \mathbb{Z}$ and also,
$0 \le a,a' < 1 \Rightarrow -1 < a-a' < 1 \Rightarrow a=a'$.
Similarly $b=b' \Rightarrow w=w'$. $\square$

4/26 f meromorphic on $\mathbb{C}$, $f(z+1)=f(z)$, $f(z+\tau)=f(z)$, $\operatorname{Im}\tau>0$.
Fundamental parallelogram $P_0=\{z\in\mathbb{C} \mid z=a+b\tau,\ 0\le a<1,\ 0\le b<1\}$.



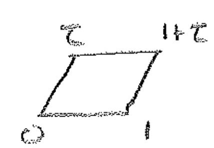
Def: A period parallelogram P is any translation of P_0.
$P=P_0+h$, $h\in\mathbb{C}$.

$*$ $\mathbb{C}=\coprod_{m,n\in\mathbb{Z}}(n+m\tau+P_0)$ (ie disjoint tiling of the plane)

Thm: An entire doubly periodic fcn is constant.
Pf: Since $\overline{P_0}$ is cpt, f bdd on P_0, but all values of f are det. by values on P_0. So f bdd & constant by Liouville's Thm. □

Def: A nonconstant doubly periodic meromorphic fcn on $\mathbb{C}$ is called an elliptic fcn.
- zeros & poles are isolated, so if f is an ell. fcn, it has only finitely many zeros & poles in P_0 (of course, has ∞ly many total)

Thm: The number of poles of an ell. fcn. in P_0 is always ≥ 2 (counted w/ multiplicity).



Pf: First, assume no poles on ∂P_0:
$$\int_{\partial P_0} f(z)\,dz = 2\pi i \sum_{\substack{\text{poles}\\ \text{in } P_0}} \operatorname{Res}(f).$$

* pole of higher order has res $=0$ *

Claim: $\int_{\partial P_0} f(z)\,dz=0$. Will prove thm: if have pole of order 1, must be 2^{nd} pole to cancel it out; if pole of order 2, res $=0$.

$$\int_{\partial P_0} f(z)\,dz = \int_0^1 f(z)\,dz + \int_1^{1+\tau} f(z)\,dz + \int_{1+\tau}^{\tau} f(z)\,dz + \int_\tau^0 f(z)\,dz$$

e.g.: $\int_0^1 f(z)\,dz + \int_{1+\tau}^{\tau} f(z)\,dz$ $\quad \downarrow s = z-\tau$

$$= \int_0^1 f(z)\,dz + \int_1^0 f(s+\tau)\,ds = \int_0^1 f(z)\,dz + \int_1^0 f(z)\,dz = 0.$$

similarly for other 2 $\int$'s

Parallel sides cancel e/o out, & $\int_{\partial P_0} f(z)\,dz = 0$, completing the proof if $\nexists$ poles on ∂P_0.

If f has a pole on ∂P_0, choose $h \in \mathbb{C}$ s.t. if $P = P_0 + h$, then f has no poles on ∂P. (can do this b/c only countably many poles in $\mathbb{C}$). By above pf, f has at least 2 poles in P & so ≥ 2 poles in P_0. $\square$

<u>Def</u>: The total number of poles in P_0 of an ell. fcn is called its <u>order</u>. (always ≥ 2)

<u>Thm</u>: Every ell. fcn of order m has precisely m zeros in P_0.

<u>Pf</u>: Assume f has no zeros or poles on ∂P_0. We know $\int_{\partial P_0} \frac{f'(z)}{f(z)}\,dz = 2\pi i(\#\text{ of zeros} - \#\text{ of poles})$

But if f has pds 1 & τ, so does f', & thus so does f'/f. Thus $\int_{\partial P_0} f'/f\,dz = 0. \Rightarrow \#$ of zeros $=$ $\#$ of poles.

$\uparrow$ no poles on ∂P_0 b/c f has no zeros on ∂P_0

If has a zero or a pole on ∂P_0, then apply this argument to $P = P_0 + h$ s.t. f has no zeros or poles on ∂P. $\square$

$F(z) = f(z) - c$. F has pds 1 & τ, & all same poles as f, & therefore the same order. The $\#$ of zeros of F in P_0 is equal to the order of $f \Rightarrow \#$ of sol'ns to $f(z) = c$ is $=$ to the order of $f \geq 2$.

So f attains every value on the fund. ~~pl~~ & attains each the order of f times.

Weierstrass $\wp$-fcn (\wp)

Let's start by constructing a fcn which has pd 1 & poles at the integers. (ex: $\cot \pi z$)

$F(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z+n}$. $F(z+1) = \sum_{n=-\infty}^{\infty} \frac{1}{z+1+n} = F(z)$. But Σ doesn't conv. abs.

$\frac{(-1)^n}{n}$ conv, but not abs.

To make this converge, define

$F(z) = \lim_{N\to\infty} \sum_{|n|\le N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \left[\frac{1}{z+n} + \frac{1}{z-n}\right]$ ← paired ⊕ n w/ ⊖ n.

Lim. is symm. in N.

$\frac{1}{z+n} + \frac{1}{z-n} = \frac{z-n+z+n}{(z+n)(z-n)} = \frac{2z}{(z+n)(z-n)}$ for n large, looks like $\frac{1}{n^2}$, which conv.

So $\sum_{n=1}^{\infty}\left(\frac{1}{z+n} + \frac{1}{z-n}\right)$ conv. abs. & unif. on cpt subsets of $\mathbb{C}\setminus\mathbb{Z}$.

If conv. abs. doesn't matter the order you take sum - if conv, but not abs, does matter the order

Equivalent way: $\frac{1}{z} + \sum_{n\neq 0}\left(\frac{1}{z+n} - \frac{1}{n}\right)$ ← terms where $n>0$ cancel w/ terms where $n<0$ conv. absolutely b/c $O(\frac{1}{n^2})$

$$\| \lim_{N\to\infty} \sum_{\substack{n\neq 0\\ |n|\le N}} \left(\frac{1}{z+n} - \frac{1}{n}\right) \qquad (\pi\cot\pi z)$$

$\frac{1}{z+n} + \frac{1}{z-n} = \left(\frac{1}{z+n} - \frac{1}{n}\right) + \left(\frac{1}{z-n} - \frac{1}{-n}\right)$, so sums are same.

Now for 2-dims: $\Lambda = \{n+m\tau \mid n,m\in\mathbb{Z}\}$

$\sum_{\omega\in\Lambda} \frac{1}{(z+\omega)^2}$

$\omega_0\in\Lambda$: $\sum_{\omega\in\Lambda} \frac{1}{(z+\omega_0+\omega)^2} = \sum_{\omega\in\Lambda}\frac{1}{(z+\omega)^2}$, b/c Λ an add gp, so this is periodic of pds 1 & τ. But doesn't conv. b/c 2-dim sum of $O(\frac{1}{\omega^2})$.

Let $\Lambda^* = \Lambda\setminus\{0\}$.

$$\frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$$

$$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{\omega^2-(z+\omega)^2}{(z+\omega)^2\omega^2} = \frac{-z^2-2z\omega}{(z+\omega)^2\omega^2} \le \frac{1}{|\omega|^3}$$

Lemma: The two series $\sum_{(n,m)\neq 0} \frac{1}{(|n|+|m|)^r}$ & $\sum_{\substack{n+m\tau \\ \in\Lambda}} \frac{1}{|n+m\tau|^r}$ conv. for $r>2$.

Pf: 1st Sum:

$n\neq 0$: $\sum_{m\in\mathbb{Z}} \frac{1}{(|n|+|m|)^r} = \frac{1}{|n|^r} + 2\sum_{m\geq 1}\frac{1}{(|n|+m)^r}$ $\qquad k=|n|+m$

$= \frac{1}{|n|^r} + 2\sum_{k\geq |n|+1} \frac{1}{k^r} \leq \frac{1}{|n|^r} + 2\int_{|n|}^{\infty} \frac{dx}{x^r} \leq \frac{1}{|n|^r} + C\frac{1}{|n|^{r-1}}$

bound Σ by $\int$

$\sum_{(n,m)\neq 0} \frac{1}{(|n|+|m|)^r} = \sum_{|m|\neq 0} \frac{1}{|m|^r} + \sum_{|n|\neq 0}\sum_{m\in\mathbb{Z}} \frac{1}{(|n|+|m|)^r}$

$\leq \underbrace{\sum_{|m|\neq 0} \frac{1}{|m|^r}}_{r>2} + \sum_{|n|\neq 0}\Big(\underbrace{\frac{1}{|n|^r}}_{r>2} + C\underbrace{\frac{1}{|n|^{r-1}}}_{r-1>1}\Big) < \infty \Rightarrow$ conv.

$\tau = a+ib$, $b>0$

$|n+m\tau| = |(n+am)+ibm| \approx |n+am| + |bm| \approx |n|+|m|$

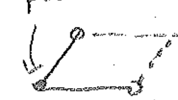
if n big $\approx n$ $\quad$ if m big $\approx m$

4/29 $\Lambda = \{m+n\tau \mid m,n\in\mathbb{Z}\}$.

$\mathcal{P}(z) \overset{?}{=} \sum_{\omega\in\Lambda} \frac{1}{(z+\omega)^2}$, $\quad \mathcal{P}(z+\omega_0) = \mathcal{P}(z)$ $\forall\, \omega_0\in\Lambda$.

- double pole at each lattice pt.

only pole in fund. ▱.



$\Lambda^* = \Lambda\setminus\{0\}$

$\frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right)$ conv. unif on cpt sets on $\mathbb{C}\setminus\Lambda$.

$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{-z^2-2z\omega}{(z+\omega)^2\omega^2} = O\left(\frac{1}{|\omega|^3}\right)$ $\quad 3>2$, so sum converges.

Pf of Lemma (ctd):

Notation: $A\approx B$ if $\exists C$ w/ $\frac{1}{C}A \leq B \leq CA$. We will show $|n|+|m| \approx |n+m\tau|$, & then sums conv. & div. together.

$\tau\in\mathbb{C}$, $\operatorname{Im}(\tau)>0$; constants will depend on τ.

Note: $(A^2+B^2)^{1/2} \approx A+B$, $\quad A,B>0$.

b/c $A,B \leq (A^2+B^2)^{1/2}$ & $(A+B)^2 \geq A^2+B^2$

$\tau = s+it$, $t>0$.

$|n+m\tau| = ((n+ms)^2 + (mt)^2)^{1/2} \approx |n+ms| + |mt|$

$\approx |n+ms| + |m|$. 2 cases: either $|n|\leq 2|m||s|$ or $|n|> 2|m||s|$. Either way, $\approx |n|+|m|$ $\square$

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right) = \frac{1}{z^2} + \sum_{(n,m)\neq(0,0)}\left(\frac{1}{(z+n+m\tau)^2} - \frac{1}{(n+m\tau)^2}\right)$$

We need this sum to conv. unif on cpt sets of $\mathbb{C}\setminus\Lambda$.
For $|z|<R$,

$$\mathcal{P}(z) = \underbrace{\frac{1}{z^2} + \sum_{|\omega|\le 2R}\left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right)}_{\text{finite sum of rat'l fcns - poles at lattice pts } \le 2R.} + \underbrace{\sum_{|\omega|>2R}\left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right)}_{\text{need this to conv. unif. on cpt sets on } |z|\le R.}$$

The term in the 2nd sum is:

$$\left|\frac{-z^2-2z\omega}{(z+\omega)^2\omega^2}\right| \le C\cdot\frac{1}{\omega^3} = C\,\frac{1}{(n+m\tau)^3}$$

$\hookleftarrow$ conv. & doesn't depend on z

So this conv. unif. for $|z|\le R$. & is holo. for $|z|\le R$.
$\mathcal{P}$ is meromorphic w/ double poles at the lattice poles.

<u>Thm</u>: The function $\mathcal{P}$ is an elliptic fcn w/ pds 1 & τ, with double poles at the lattice pts.

<u>Pf</u>: $\mathcal{P}'(z) = -2\sum_{\omega\in\Lambda}\frac{1}{(z+n+m\tau)^3}$ conv. unif. on cpt sets of $\mathbb{C}\setminus\Lambda$ (b/c $\mathcal{P}$ did)

$\mathcal{P}'$ clearly has pds 1 & τ.

$\frac{d}{dz}(\mathcal{P}(z+1) - \mathcal{P}(z)) = 0 \Rightarrow \mathcal{P}(z+1) = \mathcal{P}(z) + a$. Similarly,

$\mathcal{P}(z+\tau) = \mathcal{P}(z) + b$

$\mathcal{P}(z)$ is even: $\frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right)$ replace z w/ $-z$, then replace ω w/ $-\omega$ & get same sum.

$\Rightarrow \mathcal{P}(z) = \mathcal{P}(-z) \Rightarrow \mathcal{P}(-1/2) = \mathcal{P}(1/2) \Rightarrow a = 0$.

and $\mathcal{P}(-\tau/2) = \mathcal{P}(\tau/2) \Rightarrow b = 0$

So $\mathcal{P}$ is elliptic. $\square$

- any rat'l fcns of ell. fcns are ell.

$1/2$, $\tau/2$, $1+\tau/2$ are <u>half-periods</u>.

$\mathcal{P}'(1/2) = -\mathcal{P}'(-1/2) = -\mathcal{P}'(-1/2+1) = -\mathcal{P}'(1/2) \Rightarrow \mathcal{P}'(1/2) = 0$

$\mathcal{P}'(\tau/2) = 0$ & $\mathcal{P}'(1+\tau/2) = 0$ b/c $\mathcal{P}'$ odd (deriv of even fcn)

$\wp$ has one pole of order 2 in fund. $\wp$. $\Rightarrow$
$\wp'$ is an ell. fcn of order 3 $\Rightarrow$ precisely 3 zeros
so $1/2, \tau/2, 1+\tau/2$ are all the zeros of $\wp'$ & they are of order 1.
Define $\wp(1/2) = e_1$, $\wp(\tau/2) = e_2$, $\wp(1+\tau/2) = e_3$.
If $g(z) = \wp(z) - e_1$, then g has a root at $1/2$. But $g'(1/2) = \wp'(1/2) = 0$, so g has a double root at $1/2$.
But $\wp(z) = e_1$ has precisely 2 solutions ($\wp$ of order 2), so there are no other sol'ns, $z = 1/2$ a root of order 2.
Same for $\tau/2$, $1+\tau/2$.
e_1, e_2, e_3 are distinct (else the eqns $\wp(z) = e_i$ would have >2 sol'ns).
$\wp(a) = f$. but $\wp'(a) \neq 0$, so this only 1 sol'n, so must be another one: $\wp(-a) = f$, & the only way $a \equiv -a$ is if $a = \frac{1}{2}, \frac{\tau}{2}$, or $\frac{1+\tau}{2}$.

<u>Thm</u>: The fcn $(\wp')^2$ is given by
$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$.
<u>Pf</u>: The only roots of $F(z) = (\wp - e_1)(\wp - e_2)(\wp - e_3)$ in the fund. $\wp$ have mult 2 & are at $1/2, \tau/2, 1+\tau/2$.
Similarly for $(\wp')^2$.
$F(z)$ has a pole of order 6 at the lattice pts & nowhere else. Similarly for $(\wp')^2$.
$\frac{(\wp')^2}{F}$ has no zeros & no poles, so an entire doubly periodic fcn $\Rightarrow$ constant.
$\Rightarrow (\wp')^2 = CF$.
$\wp(z) = \frac{1}{z^2} + \text{h.o.t.}$
$\wp'(z) = -\frac{2}{z^3} + \text{h.o.t.}$
on RHS, get $\frac{C}{z^6}$, on LHS get $\frac{4}{z^6}$ $\Rightarrow C = 4$. $\square$

Thm 1: Every even ell. fcn w/ pds 1 & τ is a rat'l fcn of $\wp$.
(characterizes all even ell. fcns)

Thm 2: Every ell fcn w/ pds 1 & τ is a rat'l fcn of $\wp$ & $\wp'$.
(never need $(\wp')^n$, $n \neq 1$)

Pf (given thm 1): f an ell. fcn.
$f = f_{even} + f_{odd}$ both parts doubly pd w/ pds 1 & τ
Thm 1 applies to f_{even}, & applies to $\left.\frac{f_{odd}}{\wp'}\right\}$ an even ell. fcn.

$\square$

5/1 Exam Review

① Riemann Mapping Thm (p.229): Given any simply ctd region Ω, which is not all of $\mathbb{C}$, and $z_0 \in \Omega$, $\exists!$ analytic $f: \Omega \to \{z \mid |z|<1\}$ w/ f bijective & $f(z_0)=0$, $f'(z_0)>0$.
- Why simply ctd?
- Why do we need $\Omega \neq \mathbb{C}$?
- Know the entire proof (will have to do some part of the proof on exam)

② Schwartz's Lemma: If $f(z)$ is analytic for $|z|<1$ & satisfies $|f(z)| \leq 1$, $f(0)=0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some z or $|f'(0)|=1$, then $f(z) = cz$, $|c|=1$.
- Automorphisms of disk, $f: \{z \mid |z|<1\} \to \{z \mid |z|<1\}$ bijectively & holomorphic. $f(z) = e^{i\theta} \frac{z-a}{1-\bar{a}z}$, $|a|<1$
↳ was a HW problem - know why characterizes autos of $\mathbb{D}$

③ Phragmén-Lindelöf: Let F holo. in a sector S whose vertex is at the origin & forms an ∢ of π/β, and continuous on $\bar{S}$ with $|F(z)| \leq 1$ on ∂S, & $|F(z)| \leq Ce^{c|z|^\alpha}$, $C, c > 0$, $0 < \alpha < \beta$. Then $|F(z)| \leq 1$ in S.

- the bound is necessary. Ex: 

$F(z) = e^{z^2}$
$|F(z)| = 1$ on ∂S
but $\to \infty$ as $x \to \infty$

- need to apply it in a problem w/o P-L being mentioned.

④ Fourier Transform: Main idea → f cptly supported, (or decays fast enough)

$\hat{f}(\xi) = \int f(x) e^{-2\pi i x \xi}\, dx$ is entire

makes sense for $\xi \in \mathbb{C}$ b/c exp of e is real & can't have bounds of $\int \to \infty$.

- holo. by Morera – switch $\int$'s.
- see an integral like this w/ $\xi \in \mathbb{R}$, try changing it to a complex # where $\int$ makes sense.

⑤ Elliptic Functions: Meromorphic fcn on $\mathbb{C}$. f is doubly periodic if $f(z + \omega_1) = f(z)$, $f(z + \omega_2) = f(z)$, for some $\omega_2/\omega_1 \in \mathbb{R}$. $\tau = \omega_2/\omega_1$, $F(z) = f(\omega_1 z)$.

F has pds 1 & τ. $\mathrm{Im}(\tau) > 0$ by possibly reversing the roles of ω_1 & ω_2. P_0 = fundamental ▱ 

Know the idea of this proof:

Thm: The total # of poles in P_0 of an ell. fcn is always ≥ 2.

Pf: First assume that there are no poles on P_0. By the residue thm, $\int_{\partial P_0} f(z)\, dz = 2\pi i \sum_{\substack{\text{poles} \\ \in P_0}} \mathrm{Res}\, f$

We will show $\int_{\partial P_0} f(z)\, dz = 0$. If only one pole w/ order 1, then $\sum \mathrm{Res}\, f \neq 0$, which contradicts $\int = 0$. If no poles, f is constant.

$$\int_{\partial P_0} f(z)\, dz = \underbrace{\int_0^1 f(z)\, dz}_{①} + \int_1^{1+\tau} f(z)\, dz + \underbrace{\int_{1+\tau}^{\tau} f(z)\, dz}_{②} + \int_\tau^0 f(z)\, dz$$

$$\int_0^1 f(z)\, dz + \int_{1+\tau}^{\tau} f(z)\, dz = \int_0^1 f(z)\, dz + \int_1^0 f(s+\tau)\, ds$$
$$= \int_0^1 f(z)\, dz + \int_1^0 f(z)\, dz = 0.$$ Sim. for LHS & RHS $\Rightarrow \int_{\partial P_0} f(z)\, dz = 0$

If f has a pole on ∂P_0, choose $h \in \mathbb{C}$ so that if $P = P_0 + h$, then f has no poles on ∂P. As before, there must be at least 2 poles of f in P. Since every pt in P is congruent modulo Λ to a unique pt in P_0, there must be at least 2 poles in P_0.

- If you want to count smth w/ ell curves, write it as an $\int_{\partial P_0}$. will always get 0 for $\int$, done. //

<u>Thm 1</u>: Every ell. fcn f w/ pds 1 & τ is a rational fcn of $\wp$ & $\wp'$.

<u>Thm 2</u>: Every even ell fcn f w/ pds 1 & τ is a rational fcn of $\wp$.

<u>Pf of Thm 1 given Thm 2</u>: $f_{even} = \frac{f(z)+f(-z)}{2}$, doubly per. w/ 1 & τ
$f_{odd} = \frac{f(z)-f(-z)}{2}$, also doubly per w/ 1 & τ.
$f(z) = f_{even}(z) + f_{odd}(z)$.
f_{even} is a rat'l fcn of $\wp$ by thm 2.
$\frac{f_{odd}(z)}{\wp'(z)}$ is mero. & per. w/ 1 & τ, quotient of 2 odd fcns $\Rightarrow$ even doubly per w/ 1 & τ, so thm 2 applies to see this as a rat'l fcn in $\wp$.
That rat'l fcn times $\wp'(z)$ is f_{odd} & is rat'l fcn in $\wp$ & $\wp'$.

<u>Note</u>: Only ever need one $\wp'$. (clear b/c $(\wp')^2$ can be written w/ $\wp$)

<u>Recall</u>: 3 half-pds: $\tau/2$, $1+\tau/2$, $1/2$. (Result will hold for all 3).
Let f be an odd ell fcn.
$\underbrace{f(-\tau/2) = -f(\tau/2)}_{+1} = f(\tau/2) \Rightarrow f(\tau/2) = 0$

Suppose f is an even ell. fcn w/ a zero at $\tau/2$,

- cannot be zero of order 1 b/c $f'(\tau/2)=0$.
- Can't be a zero of order 3 b/c $f'''(\tau/2)=0$
  (apply above to $f''(z)$, even).
- a zero at a half-pd must have <u>even</u> order
  (for an even ell. fcn)

If $a\in P_0$ is not a half pd, f even, but $f(a)=0$, then $f(-a)=f(a)=0$. But $-a\sim b\in P_0$ for some $b\neq a$ in P_0, b/c the only way $-a\sim a$ is if a is a half period. So $f(b)=0$; & orders of zero at a & b are equal.

<u>Preview</u>: Start w/ 1 & τ, make $\wp_\tau(z)$.
If we change $\underbrace{\tau\mapsto\tau+1}_{\text{doesn't change pds}}$, then $\wp_{\tau+1}(z)=\wp_\tau(z)$

$\tau=\omega_2/\omega_1\longmapsto -\frac{1}{\tau}=-\omega_1/\omega_2 \Rightarrow \wp_{-1/\tau}(z)=\tau^2\wp_\tau(\tau z)$

modular gp: the gp of these transformations.

5/3 Weierstrass $\wp$ fcn. Given $1,\tau$, $\operatorname{Im}(\tau)>0$
$\wp_{\tau+1}(z)=\wp_\tau(z)$, $\wp_{-\frac{1}{\tau}}(z)=\tau^2\wp_\tau(\tau z)$

<u>Eisenstein Series</u>
$E_k(\tau)=\sum_{(n,m)\neq(0,0)}\frac{1}{(n+m\tau)^k}$, k is an integer ≥ 3
$=\sum_{\omega\in\Lambda^*}\frac{1}{\omega^k}$

<u>Thm</u>: (i) $E_k(\tau)$ conv. if $k\geq 3$ and is holomorphic in the upper half plane.
(ii) $E_k(\tau)=0$ if k is odd
(iii) $E_k(\tau+1)=E_k(\tau)$; $E_k(\tau)=\tau^{-k}E_k(-1/\tau)$

<u>Pf</u> (i) If $\operatorname{Im}(\tau)\geq\delta>0$, $\sum_{(n,m)\neq(0,0)}\frac{1}{(n+m\tau)^k}$ conv abs. & uniformly, so conv. unif. on cpts in the upper half plane & defines a holo. fcn in the upper half plane

Need $\operatorname{Im}(\tau)>0$: $\tau=a+ib$: $|n+m\tau|=|(n+am)+ibm|\approx|n+am|+|bm|$
need $b>0$ b/c if n big⊕, m big⊖, first term could have cancellation, but then $|bm|$ is big.

(ii) $\sum \frac{1}{(n+m\tau)^k} = \sum \frac{1}{(-n-m\tau)^k} = -\sum \frac{1}{(n+m\tau)^k}$ if k odd,

so $= 0$ for odd k.

(iii) $\sum \frac{1}{(n+m(\tau+1))^k} = \sum \frac{1}{(n+m\tau+m)^k}$, just change indices

$\quad$ & $= E_k(\tau)$,

$$\sum \frac{1}{(n+m(-1/\tau))^k} = \sum \frac{1}{\tau^{-k}(n\tau-m)^k} = \tau^k \sum \frac{1}{(n\tau-m)^k} = \tau^k E_k(\tau)$$

<u>Thm</u>: For z near 0, $\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) E_{2k+2}\, z^{2k}$

$$= \frac{1}{z^2} + 3E_4 z^2 + 5E_6 z^4 + \cdots$$

<u>Pf</u>: $\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*} \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]$

$$= \frac{1}{z^2} + \sum_{\omega\in\Lambda^*} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$$

For $|\omega|<1$, $\frac{1}{1-\omega} = \sum \omega^\ell$. Differentiating, we see

$$\frac{1}{(1-\omega)^2} = \sum_{\ell=1}^{\infty} (\ell+1)\omega^\ell$$

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2}\cdot\frac{1}{(\frac{z}{\omega}-1)^2} = \frac{1}{\omega^2}\sum_{\ell=0}^{\infty}(\ell+1)\left(\frac{z}{\omega}\right)^\ell \qquad (\text{ok, b/c } |z|<|\omega|)$$

$$= \frac{1}{\omega^2} + \frac{1}{\omega^2}\sum_{\ell=1}^{\infty}(\ell+1)\left(\frac{z}{\omega}\right)^\ell$$

abs conv, so Fubini can change Σ's

$$\Rightarrow \wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\sum_{\ell=1}^{\infty}(\ell+1)\frac{z^\ell}{\omega^{\ell+2}} = \frac{1}{z^2} + \sum_{\ell=1}^{\infty}(\ell+1)\Big(\sum_{\omega\in\Lambda^*}\frac{1}{\omega^{\ell+2}}\Big) z^\ell$$

$$= \frac{1}{z^2} + \sum_{\ell=1}^{\infty}(\ell+1)\, E_{\ell+2}\, z^\ell$$, & $E_{\ell+2} = 0$ when $\ell+2$ odd, so $\ell = 2k$

$$= \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)E_{2k+2}\, z^{2k}$$

$\wp(z) = \frac{1}{z^2} + 3E_4 z^2 + 5E_6 z^4 + \cdots$

$\wp'(z) = -\frac{2}{z^3} + 6E_4 z + 20E_6 z^3 + \cdots$

$(\wp'(z))^2 = \frac{4}{z^6} - \frac{24E_4}{z^2} - 80E_6 + \cdots$

$\wp(z)^3 = \frac{1}{z^6} + \frac{9E_4}{z^2} + 15E_6 + \cdots$

$(\wp'(z))^2 - 4\wp(z)^3 + 60E_4\wp(z) + 140E_6$ is holomorphic near zero & zero at zero (b/c all $\frac{1}{z}$'s cancel, & so does constant)

an ell fcn has a pole only at 0 in P_0, & this doesn't.

So this is an entire ell. fcn & must be constant, & since $=0$ at 0, equals 0 everywhere. So we've written $(\wp'(z))^2$ as a poly in $\wp(z)$, as before.

Cor: If $g_2 = 60E_4$ & $g_3 = 140E_6$, then
$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

The divisor fcn:
$$\sum_{d|r} d^\ell = \sigma_\ell(r)$$

Thm: If $k \geq 4$ is even & $\operatorname{Im}\tau > 0$, then
$$E_k(\tau) = 2\zeta(k) + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \cdot \sum_{r=1}^{\infty} \sigma_{k-1}(r)e^{2\pi i\tau r}$$

Pf: $E_k(\tau) = \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^k} = \sum_{\substack{n\neq 0 \\ (\& m=0)}} \frac{1}{n^k} + \sum_{m\neq 0}\sum_{n=-\infty}^{\infty} \frac{1}{(n+m\tau)^k}$

$= 2\zeta(k) + \sum_{m\neq 0}\sum_{n=-\infty}^{\infty} \frac{1}{(n+m\tau)^k} = 2\zeta(k) + 2\sum_{m>0} \frac{1}{(n+m\tau)^k}$ (b/c k even)

$= 2\zeta(k) + 2\sum_{m>0} \frac{(-2\pi i)^k}{(k-1)!} \sum_{\ell=1}^{\infty} \ell^{k-1} e^{2\pi i m\tau\ell}$

by Poisson summation : $\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \hat{f}(k)$

- apply to $\frac{1}{(z+\tau)^k}$

from i b/c k even

$= 2\zeta(k) + \frac{2(-1)^{k/2}(2\pi)^k}{(k-1)!} \cdot \sum_{m>0}\sum_{\ell=1}^{\infty} \ell^{k-1} e^{2\pi i m\ell}$

$\sum_{m>0}\sum_{\ell=1}^{\infty} \ell^{k-1} e^{2\pi i m\ell\tau} = \sum_{r=1}^{\infty} \sigma_{k-1}(r)e^{2\pi i r\tau}$

fix $m\ell$ as var, then ℓ goes over divisors of $m\ell$

5/10 ④ $a \in C^\infty([0,1])$, $\left|\int_0^1 a(x)e^{-\lambda x}\,dx\right| \le e^{-\lambda}$, $\lambda \ge 1$

$\cdot\, e^{\lambda}$: ↑ Laplace transform of a.

$\left|\int_0^1 a(x)e^{(1-x)\lambda}\,dx\right| \le 1$ $\quad x = 1-x$

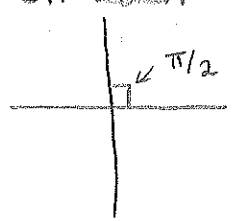
$\left|\int_0^1 \underbrace{a(1-x)}_{b(x)}e^{\lambda x}\,dx\right| \le 1$

Assume $\left|\int_0^1 b(x)e^{\lambda x}\,dx\right| \le 1$

$F(\lambda) = \int_0^1 b(x)e^{-\lambda x}\,dx$ $\quad \lambda \in \mathbb{C}$, entire.

$|F(\lambda)| \le \int_0^1 |b(x)|e^{|\lambda||x|}\,dx \le Ce^{|\lambda|}$, so can apply P-L on ∡'s less than π. By assumpton, $|F(\lambda)| \le 1$ for $\lambda \ge 1$. $|F(\lambda)| \le C$ for $\lambda \ge 0$. $|F(\lambda)| \le C$ for $\lambda \le 0$ & $|F(\lambda)| \to 0$ as $\lambda \to -\infty$ along real axis (b/c $e^{\lambda x}$ is small if $\lambda < 0$)

For $\lambda \in i\mathbb{R}$, $|e^{\lambda x}| = 1$, so $\left|\int_0^1 a(x)e^{\lambda x}\,dx\right| \le \int_0^1 |a(x)|\,dx \le C$. so bdd on each of the axes.



Apply P-L in each quadrant
Conclude $|F(\lambda)| \le C$ $\forall \lambda$.

So $F(\lambda)$ is constant, but $|F(\lambda)| \to 0$ as $\lambda \to -\infty$, so $F(\lambda) = 0$.

$\int_0^1 b(x)e^{\lambda x}\,dx = 0$ $\forall \lambda$. Many ways to get $b = 0$:
$\lambda = -2\pi i \xi$, so
$\hat{b}(\xi) = \int_0^1 b(x)e^{-2\pi i \xi x}\,dx = 0$ $\forall \xi \Rightarrow b \equiv 0$.

① $|f(z)| \le |g(z)|$. Assume $g \equiv 0$.
$h(z) = \frac{f(z)}{g(z)}$ is a meromorphic fcn w/ $|h(z)| \le 1$, so all singularities are removable $\Rightarrow$ h is an entire bdd fcn $\Rightarrow$ h constant, so $f(z) = cg(z)$, $|c| \le 1$.

② Hurwitz's Thm: If the fcns $f_n(z)$ are analytic in a region Ω & never zero, & $f_n \to f$ unif. on cpt sets, then $f \equiv 0$, or f never zero in Ω.

Pf: Suppose $f \not\equiv 0$. The zeros of f are isolated, so for $z_0 \in \Omega$, $\exists\, r>0$ s.t. $f(z) \neq 0$ on $0<|z-z_0|\leq r$. In particular, $f(z)$ has a positive min. on cpt set $|z-z_0|=r$. Since $f_n \to f$ on $|z-z_0|=r$, & since $|f| \geq c > 0$ on the same set, $\frac{1}{f_n} \to \frac{1}{f}$ unif. on $|z-z_0|=r$.

Let $C = \{|z-z_0|=r\}$.

$$0 = \lim_{n\to\infty} \frac{1}{2\pi i}\int_C \frac{f_n'(z)}{f_n(z)}\,dz \overset{DCT}{=} \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz \Rightarrow \text{\# of zeros of}$$

f in C is zero, as desired.

③ Schwartz's Lemma: If $f(z)$ is analytic for $|z|<1$ & satisfies $|f(z)| \leq 1$, $f(0)=0$, then $|f(z)| \leq |z|$ & $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some z or $|f'(0)|=1$, then $f(z) = cz$ w/ $|c|=1$.

Cor: Suppose $f: \mathbb{D} \to \mathbb{D}$, is bijective & holo. w/ $f(0)=0$. Then $f(z) = cz$ w/ $|c|=1$.

Pf: We know $|f'(0)| \leq 1$. Also, $\frac{\partial f^{-1}}{\partial z}(0) = \frac{\partial f^{-1}}{\partial z}(f(0)) = \frac{1}{f'(0)}$ so $\left|\frac{1}{f'(0)}\right| \leq 1 \Rightarrow |f'(0)| = 1$ & $\therefore$ $f(z) = cz$ w/ $|c|=1$.

Suppose Ω is a s. ctd open set & f, g are 2 bijective holo. fcns taking $\Omega \to \mathbb{D}$ w/ $z_0 \mapsto 0$. What is the relationship btwn f & g.

$f \circ g^{-1}: \mathbb{D} \to \mathbb{D}$ bijectively & $f \circ g^{-1}(0) = 0$. By the corollary $f\circ g^{-1}(z) = \underset{\uparrow\ |c|=1}{c}z$. Let $z = g(w)$ & we get $f(w) = c\,g(w)$.

⑤ A non-constant elliptic fcn f has the same # of zeros & poles.

Let P_0 be the fundamental parallelogram. First assume f has no zeros or poles on ∂P_0. We claim $\frac{1}{2\pi i}\int_{\partial P_0} \frac{f'(z)}{f(z)}\,dz = 0$, since this is the (# of zeros) − (# of poles), we're done. Let $g(z) = f'(z)/f(z)$. g is an ell fcn of pds 1 & τ.

$$\int_{\partial P_0} g(z)dz = \int_0^1 g(z)dz + \int_1^{1+\tau} g(z)dz + \int_{1+\tau}^{\tau} g(z)dz + \int_\tau^0 g(z)dz$$

$$\int_1^{1+\tau} g(z)dz + \int_0^1 g(z)dz = \int_0^1 g(z)dz + \int_1^0 \underset{\overset{\parallel}{g(z)}}{g(z+\tau)}\,dz$$

$$= \int_0^1 g(z)dz - \int_0^1 g(z)dz = 0.$$ Similarly for other pair.

If f has a zero or pole on ∂P_0, translate P_0 by some $h \in \mathbb{C}$ so that if $P = P_0 + h$, then f doesn't have any zeros or poles on ∂P. Every pt in P_0 is $\sim$ to precisely one pt in P, so f has the same # of zeros & poles in P.

How #4 came up in research: $f \in C^\infty$

$f(t,x): [0,1] \times [0,1] \to \mathbb{R}$.

$$\partial f/\partial t\,(x,t) = \begin{cases} \dfrac{f(x,t) - f(0,t)}{x} & x > 0 \\ \dfrac{\partial f}{\partial x}(0,t), & x = 0 \end{cases}$$ (comes up in medical imaging)

Uniqueness? Suppose f & g satisfy the above where $f(x,0) = g(x,0)$ $\forall x$. Does $f = g$? Yes.

$h = f - g$, h sat. the above, & $h(x,0) = 0$. Goal; $h \equiv 0$.

$$\frac{\partial h}{\partial t}(x,t) = \frac{h(x,t) - \cancel{h(0,t)}\,a(t)}{x}$$

$$h(x,t) = -\int_0^t e^{(t-s)/x}\,\frac{a(s)}{x}\,ds$$ (sol'n to the lin. diff. eqn)

$h \in C^\infty$, so bdd as $x \to 0$. Call $1/x = \lambda$, apply #4

$\Rightarrow a(s) = 0$. $\Rightarrow \frac{\partial h}{\partial t} = \frac{h(x,t)}{x} \Rightarrow h \equiv 0$

Open Question: $f(x,t): [0,1] \times [0,1] \to \mathbb{R}$, $f > 0$.

$$\frac{\partial f}{\partial t}(x,t) = \begin{cases} \dfrac{f(x,t)^2 - f(0,t)^2}{x}, & x > 0 \\ 2f(0,t)\dfrac{\partial f}{\partial x}(0,t), & x = 0 \end{cases}$$

Uniqueness?