1. Suppose E is an extension field of F. If $\alpha \in E$ is algebraic over F and $\beta \in F(\alpha)$, prove that $\deg(\beta, F)$ divides $\deg(\alpha, F)$.

2. Suppose E is a finite extension of a field F and $[E : F]$ is a prime number. Prove that $E = F(\alpha)$ for every $\alpha \in E$ which is not already in F.

3. Prove that the extensions $\mathbb{Q}(\sqrt{2} + \sqrt{5})$ and $\mathbb{Q}(\sqrt{2}, \sqrt{5})$ of $\mathbb{Q}$ are equal.

4. Suppose I and J are ideals of a ring R. Let
$$I + J = \{a + b \mid a \in I, b \in J\}.$$
Prove that $I + J$ is an ideal of R.

5. Let G be a group, and let $a, b \in G$ be elements such that $|a|$ and $|b|$ are relatively prime. Prove that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

6. Let $\phi \colon R \to R'$ be a surjective ring homomorphism. Prove that if S is a subring of R, then $\phi(S)$ is subring of R'.

7. Consider the ring $\mathbb{Z}[x]$. Prove using the definitions that the element x is neither a unit nor a zero divisor.

8. Let p, q be prime numbers. Find all elements of $\mathbb{Z}_p \times \mathbb{Z}_q$ which are their own multiplicative inverses (i.e., that are solutions to the equation $x^2 - 1 = 0$), and justify your answer.

9. Suppose R is a ring with unity. If R has finite characteristic n and n is not prime, prove (from definitions) that R has zero divisors.

10. Suppose R is a ring with no zero divisors, and let $a, b, c \in R$. Prove that the left cancellation law holds, i.e., that if $ab = ac$ and $a \neq 0$, then $b = c$.

11. Suppose S and T are subrings of a ring R. Prove that $S \cap T$ is a subring of R. What if S and T are ideals – is $S \cap T$ also an ideal?

12. Determine, with proof, the isomorphism type of $\mathbb{Z}_6 \times \mathbb{Z}_8 / \langle (0, 2) \rangle$, using the fundamental theorem of finitely generated abelian groups.

13. Determine, with proof, the isomorphism type of $\mathbb{Z} \times \mathbb{Z} / \langle (1, 1) \rangle$, using the fundamental theorem of finitely generated abelian groups.

14. Let G be a group with $|G| > 1$ and suppose G has no nontrivial proper subgroups. Show that G is a finite cyclic group of prime order.

15. Show that the alternating group A_4 contains a subgroup isomorphic to the Klein-4 group.

16. Let G be a group, and let $Z(G)$ be the center of G. Show that if $G/Z(G)$ is cyclic, then G is abelian.

17. Show that if G is a finite group such that for all $g \in G$ we have $g^2 = e$, then

    (a) G is abelian,

    (b) $|G| = 2^n$ for some positive integer n,

    (c) $G \simeq \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

18. A subgroup H of a group G is called a **characteristic subgroup** of G if for all $\phi \in \mathrm{Aut}(G)$ we have $\phi(H) = H$. (Here, $\mathrm{Aut}(G) = \{\phi\colon G \to G \mid \phi \text{ is an isomorphism}\}$, the *automorphism group* of G.)

    (a) Show that if H is a characteristic subgroup of G, then $H \triangleleft G$.

    (b) Let G be a group, H a normal subgroup of G, and K a characteristic subgroup of H. Show that K is a normal subgroup of G.

19. Let a be a nilpotent element in a commutative ring R with unity (that is, there exists $n \in \mathbb{Z}$ such that $a^n = 0$). Show that

    (a) $a = 0$ or a is a zero divisor.

    (b) ax is nilpotent for all $x \in R$.

    (c) $1 + a$ is a unit in R.

    (d) If u is a unit in R, then $u + a$ is also a unit in R.

20. Let $I = n\mathbb{Z}$ and $J = m\mathbb{Z}$ be two ideals in $\mathbb{Z}$. Then $I + J = k\mathbb{Z}$ for some $k \in \mathbb{Z}$. Express k in terms of n and m.

21. Let R be a commutative ring. The **annihilator** of R is defined as follows:

$$\mathrm{Ann}(R) = \{a \in R \mid ax = 0 \text{ for all } x \in R\}.$$

    Show that $\mathrm{Ann}(R)$ is an ideal of R.

22. Let R be a commutative ring and I and ideal in R. The **radical** of I is defined as

$$\mathrm{rad}(I) = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{Z}\}.$$

    Show that $\mathrm{rad}(I)$ is an ideal of R containing I.

23. (a) Prove that $\mathbb{R} \times \mathbb{R}$ and $\mathbb{C}$ are isomorphic groups.

    (b) Prove that $\mathbb{R}^* \times \mathbb{R}^*$ and $\mathbb{C}^*$ are not isomorphic groups.

24. Prove there is no group isomorphism from $\mathbb{Z}_8 \times \mathbb{Z}_2$ to $\mathbb{Z}_4 \times \mathbb{Z}_4$.