Math 113 | **Final Exam Review: Examples** | Spring
Section 4 | **Solutions** | 2018

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Give a non-trivial example of each of the following objects. All are possible. If you can come up with more than one example of some, that would be good practice.

1. A finite ring of matrices: $M_2(\mathbb{Z}_2)$
2. A subring of $\mathbb{C}$ which does not have unity: $2\mathbb{Z}$
3. A matrix ring which does not have unity: $M_2(2\mathbb{Z})$
4. Three units in $M_3(\mathbb{Z})$: $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$
5. A ring homomorphism $\phi : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$: $\phi(n) = (n, 0)$
6. A finite ring with at least 3 zero divisors: $\mathbb{Z}_{10}$
7. A ring whose only units are 1 and $-1$: $\mathbb{Z}$
8. A solution of the equation $x^2 + 5x + 6$ in $Z_{12}$ other than $-2$ or $-3$ (which come from factoring): 1
9. A finite fields with at least 20 elements: $\mathbb{Z}_{23}$
10. A zero divisor of the ring $\mathbb{Z}_5 \times \mathbb{Z}_7$: $(0, 1)$
11. A polynomial ring which is an integral domain: $\mathbb{Z}[x]$
12. A ring without unity that has no zero divisors: $2\mathbb{Z}$
13. A polynomial ring which is not an integral domain: $\mathbb{Z}_6[x]$
14. An integral domain whose field of quotients is $\mathbb{R}$: $\mathbb{R}$
15. An ideal of $\mathbb{Z}_3 \times \mathbb{Z}_4$ which is not a prime ideal: $((0, 0)) = \{(0, 0)\}$
16. A principal ideal of $\mathbb{Z}_3 \times \mathbb{Z}_4$ which is a prime ideal: $((0, 1))$
17. A maximal ideal of $\mathbb{R}[x]$: $(x)$
18. A ring which has no proper nontrivial maximal ideals: $\mathbb{Q}$
19. A ring $R$ which is an integral domain but not a field, and an ideal $I$ of $R$ such that $R/I$ is not a field: $R = \mathbb{Z}$, $I = (6)$

20. A ring $R$ which is an integral domain but not a field, and an ideal $I$ of $R$ such that $R/I$ is a field: $R = \mathbb{Z}$, $I = (3)$

21. A polynomial ring $R$ which is an integral domain, and an ideal $I$ of $R$ such that $R/I$ has zero divisors: $R = \mathbb{Z}[x]$, $I = (x^2 - 6x + 5)$

22. A ring $R$ which is not an integral domain, and an ideal $I$ of $R$ such that $R/I$ is an integral domain: $R = \mathbb{Z}_6$, $I = (3)$

23. Two non-isomorphic rings which each contain 3 elements: $\mathbb{Z}_3$ with the usual addition and multiplication & $(\mathbb{Z}_3, +, *)$, where $+$ is the usual addition and $*$ is defined by $a * b = 0$ for all $a, b \in \mathbb{Z}_3$.

24. A non-trivial ring homomorphism $\phi\colon \mathbb{Z}[x] \to \mathbb{Z}_3[x]$: $\phi(p(x)) = p(0) \mod 3$

25. A nontrivial ring homomorphism $\phi\colon \mathbb{Z}[x] \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$: $\phi(p(x)) = (p(0), 0, 0)$

26. A polynomial in $\mathbb{Z}[x]$ which has 4 terms and is irreducible over $\mathbb{Q}$ by Eisenstein's Criteria: $x^3 + 3x^2 + 3x + 3$

27. A polynomial in $\mathbb{Z}[x]$ which has 4 terms and is irreducible over $\mathbb{Q}$, but Eisenstein's Criteria do not apply: $x^4 + 3x^2 + 3x + 9$

28. An irreducible quadratic polynomial in $\mathbb{Z}_5[x]$: $x^2 + x + 1$

29. Two different proper subgroups $A$ and $B$ of $D_4$ such that $A \triangleleft B$ and $B \triangleleft D_4$, but $A$ is not a normal subgroup of $D_4$: $A = \langle \tau \rangle$, $B = \langle \tau, \rho^2 \rangle$ (Note: this was a homework problem.)

30. Two subgroups $A$ and $B$ of $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ such that $G/A \simeq \mathbb{Z}_4$ and $G/B \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$: $A = \langle (1, 0) \rangle$, $B = \langle (0, 2) \rangle$

31. A non-trivial homomorphism $\phi\colon \mathbb{Z}_{12} \times D_4 \to \mathbb{Z}_4$: $\phi((a, b)) = a \mod 4$

32. A subgroup of $S_3 \times \mathbb{Z}_4$ which has exactly 8 elements: $\langle (1\,2) \rangle \times \mathbb{Z}_4$

33. An infinite group $G$ and a subgroup $H$ such that there are infinitely many left cosets of $H$ in $G$: $G = \mathbb{Z} \times \mathbb{Z}$, $H = \langle (1, 1) \rangle$

34. An element of $\mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_6$ which has order 10 and does not have a 0 in any component: $(2, 1, 3)$

35. A non-abelian group with at least 6 elements of order 5: $S_6$

36. A pair of zero divisors in the ring $\mathbb{Z}_5 \times M_2(\mathbb{Z})$: $(0, I)$ and $(1, 0)$, where $I$ is the identity matrix, and the 0 in $(1, 0)$ is the zero matrix

37. An extension of $\mathbb{Q}$ which is algebraic of degree 4: $\mathbb{Q}[x]/(x^4+3x^3+3x^2+3x+3)$. Equivalently, $\mathbb{Q}(\alpha)$ where $\alpha \in \mathbb{C}$ is a root of $x^4 + 3x^3 + 3x^2 + 3x + 3$.

38. Given an example of a commutative ring without zero-divisors that is not an integral domain: $2\mathbb{Z}$.

39. Find a ring $R$ and two elements $a, b \in R$ such that $a$ and $b$ are zero-divisors and $a + b$ is a unit: $R = \mathbb{Z}_6$, $a = 2$, $b = 3$.