① We will use the 1^{st} Isomorphism Thm to prove this.

Let $\varphi: G \to \mathrm{Inn}(G)$ be defined by $\varphi(g) = i_g$.

- well-def: let $g_1 = g_2$. Then $\forall x \in G$, $i_{g_1}(x) = g_1 x g_1^{-1} = g_2 x g_2^{-1} = i_{g_2}(x)$, so $\varphi(g_1) = i_{g_1} = i_{g_2} = \varphi(g_2)$. ✓
- hom: let $g, h \in G$. Then $\forall x \in G$, $i_g \circ i_h(x) = i_g(hxh^{-1}) = ghxh^{-1}g^{-1} = (gh)x(gh)^{-1} = i_{gh}$. Thus, $\varphi(g)\varphi(h) = i_g \circ i_h = i_{gh} = \varphi(gh)$, so φ is a hom. ✓
- surj: Let $i_g \in \mathrm{Inn}(G)$. Then $g \in G$, & $\varphi(g) = i_g$, so φ is surj. ✓
- $\mathrm{Ker}\,\varphi = \{g \in G \mid i_g \text{ is the identity in } \mathrm{Inn}(G)\}$.

  In $\mathrm{Inn}(G)$, the identity is i_e, which is defined by $i_e(x) = x \ \forall x \in G$ (this is the identity hom.).

  $\varphi(g) = i_e \Leftrightarrow i_g = i_e \Leftrightarrow \forall x \in G,\ i_g(x) = i_g(x) \Leftrightarrow \forall x \in X,\ gxg^{-1} = x$
  $\Leftrightarrow gx = gx \ \forall x \in X \Leftrightarrow g \in Z(G)$.

  Thus, $\mathrm{Ker}\,\varphi = Z(G)$.

Therefore, by the 1^{st} Isomorphism Thm, $G/\mathrm{Ker}\varphi \cong \varphi(G)$, so $G/Z(G) \cong \mathrm{Inn}(G)$.

② Since $|G_1| = 10$, $\forall a \in G_1$, $a^{10} = e_1$ (where e_1 is the identity in G_1), & since $|G_2| = 15$, $\forall b \in G_2$, $b^{15} = e_2$ (where e_2 is the identity in G_2).

Let $(a,b) \in G_1 \times G_2$. Then

$$(a,b)^{30} = (a^{30}, b^{30}) = ((a^{10})^3, (b^{15})^2) = (e_1^3, e_2^2) = (e_1, e_2), \text{ so } |(a,b)| \le 30.$$

Since (a,b) was arbitrary, every elt of $G_1 \times G_2$ has order at most 30. But $|G_1 \times G_2| = 10 \cdot 15 = 150$, so no elt of $G_1 \times G_2$ has order 150, & therefore G is not cyclic.

③ As $|G| = 27$, & $g \in G$, then by Lagrange's Thm, $|g| = 1, 3, 9$, or 27. If $|g| = 1, 3,$ or 9 then $g^9 = e$. Since $a^9 \neq e$, $|a| \neq 1, 3,$ or 9. Thus $|a| = 27$, & so $G = \langle a \rangle$.

④ Since $|G:H| = 2$, H has 2 left cosets, H & gH, & 2 right cosets, H & Hg. Since $G = H \cup gH$ & $G = H \cup Hg$, it must be that $gH = Hg$. Thus the left cosets of G are equal to the right cosets, so $H \trianglelefteq G$.

⑤ (a) Suppose $\varphi: S_3 \to \mathbb{Z}_3$ is a hom. The only normal subgps of S_3 are $\{e\}$, A_3, & S_3. We consider each in turn.

- If $\mathrm{Ker}\,\varphi = \{e\}$, then φ is injective, so $|\varphi(S_3)| = |S_3| = 6$. Also, $\varphi(S_3) \le \mathbb{Z}_3$, so $|\varphi(S_3)|$ must divide $|\mathbb{Z}_3| = 3$, by Lagrange's Thm. But 6 doesn't divide 3, which is a contradiction, so there is no hom. φ w/ $\mathrm{Ker}\,\varphi = \{e\}$.
- If $\mathrm{Ker}\,\varphi = A_3$, then $|\varphi(S_3)| = |S_3/A_3| = |S_3|/|A_3| = 6/3 = 2$. But, as above, we reach a contradiction, since 2 doesn't divide 3.
- If $\mathrm{Ker}\,\varphi = S_3$, then φ is the trivial homomorphism.

Thus, the only hom. $\varphi: S_3 \to \mathbb{Z}_3$ is the trivial hom.

(b) A map $\varphi: \mathbb{Z}_4 \to \mathbb{Z}_6$ is a hom. $\Leftrightarrow |\varphi(\bar{1})|$ divides $|\bar{1}|$, since $\mathbb{Z}_4$ is cyclic. The possibilities for $\varphi(\bar{1})$ are $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}$, & $\bar{5}$, which have orders $1, 6, 3, 2, 3,$ & 6, respectively. In $\mathbb{Z}_4$, $|\bar{1}| = 4$, & of the options for $\varphi(\bar{1})$, only $|\bar{0}| = 1$ & $|\bar{3}| = 2$ divide 4.

Thus there are 2 homs:

$\varphi(\bar{1}) = \bar{0}$ (the trivial hom) & $\varphi(\bar{1}) = \bar{3}$.

⑥ $|\mathbb{Z}_4 \times \mathbb{Z}_8 / \langle(\bar{1}, \bar{2})\rangle| = |\mathbb{Z}_4 \times \mathbb{Z}_8| / |\langle(\bar{1},\bar{2})\rangle| = 32/\mathrm{lcm}(4,4) = 32/4 = 8$

The possible isomorphism types of a gp of order 8 are $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, & $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$|(\bar{0},\bar{1})+H|$ is the smallest $k\in\mathbb{Z}^+$ s.t. $k\cdot(\bar{0},\bar{1})\in H$.

$H=\{(\bar{0},\bar{0}),(\bar{1},\bar{2}),(\bar{2},\bar{4}),(\bar{3},\bar{6})\}$

$(\bar{0},\bar{1})\notin H$

$2\cdot(\bar{0},\bar{1})=(\bar{0},\bar{2})\notin H$

$3\cdot(\bar{0},\bar{1})=(\bar{0},\bar{3})\notin H$

$\vdots \qquad \vdots$

$7(\bar{0},\bar{1})=(\bar{0},\bar{7})\notin H$

$8(\bar{0},\bar{1})=(\bar{0},\bar{0})\in H \Rightarrow |(\bar{0},\bar{1})+H|=8.$

Of the possible isomorphism types above, only $\mathbb{Z}_8$ has an elt of order 8.
Therefore, $\mathbb{Z}_4\times\mathbb{Z}_8/\langle(\bar{1},\bar{2})\rangle\cong\mathbb{Z}_8$.

⑦(a) Let $\sigma\in \mathrm{Aut}(G)$. Then $\sigma^{-1}\in\mathrm{Aut}(G)$, so $\sigma^{-1}(H)\subseteq H$.
Thus, $H=\sigma(\sigma^{-1}(H))\subseteq\sigma(H)\subseteq H \Rightarrow \sigma(H)=H$.

(b) By (a), we need to show that $\sigma(Z(G))\subseteq Z(G)$ $\forall\,\sigma\in\mathrm{Aut}(G)$. Let $\sigma\in\mathrm{Aut}(G)$.
Let $x\in\sigma(Z(G))$. Then $\exists\, y\in Z(G)$ s.t. $\sigma(y)=x$.
Let $g\in G$. $\sigma(G)=G$, so $\exists\, g'\in G$ s.t. $\sigma(g')=g$.
Then
$$\begin{aligned} xg &= \sigma(y)\sigma(g') \\ &= \sigma(yg') \\ &= \sigma(g'y) \quad \text{b/c } y\in Z(G) \\ &= \sigma(g')\sigma(y) \\ &= gx. \end{aligned}$$
Since g was arbitrary, $gx=xg$ $\forall g\in G$. Therefore $x\in Z(G)$.
Thus, $Z(G)$ is a characteristic subgp of G.

⑧ $M=\{(a_1,a_2)\in G_1\times G_2 \mid f_1(a_1)=f_2(a_2)\}$

- closed: let $(a_1,a_2),(b_1,b_2)\in M$. Then $(a_1,a_2)\cdot(b_1,b_2)=(a_1b_1,a_2b_2)$.
$$\begin{aligned} f_1(a_1b_1) &= f_1(a_1)f_1(b_1) \\ &= f_2(a_2)f_2(b_2) \\ &= f_2(a_2b_2) \Rightarrow (a_1b_1,a_2b_2)\in M. \end{aligned}$$
- id: $f_1(e_1)=e_H=f_2(e_2) \Rightarrow (e_1,e_2)\in M$ (e_i is the identity of G_i for $i=1,2$, & e_H is the identity of H).
- inverses: let $(a_1,a_2)\in M$. Then $f_1(a_1^{-1})=f_1(a_1)^{-1}=f_2(a_2)^{-1}=f_2(a_2^{-1})$
$\Rightarrow (a_1^{-1},a_2^{-1})\in M$

Therefore, M is a subgp of $G_1\times G_2$.

⑨ ($\Rightarrow$) Suppose G is cyclic & generated by $a\in G$. Consider the map $\varphi:\mathbb{Z}\to G$ def. by $\varphi(n)=a^n$. Then either $G\cong\mathbb{Z}$ (if $|G|=\infty$), or if $|G|=k<\infty$, then $G\cong\mathbb{Z}_k$. In the 1st case, we have already shown φ is a hom. In the 2nd case, this map is the map $\varphi(n)=n \pmod k$, which we have also already shown is a hom. In both cases, we showed φ was surj.

($\Leftarrow$): Suppose $\exists$ surj hom $\varphi:\mathbb{Z}\to G$, & let $a=\varphi(1)\in G$. Then since φ is surj, $\forall g\in G$
$$\begin{aligned} g &= \varphi(n) \text{ for some } n\in\mathbb{Z} \\ &= \underbrace{\varphi(1)\cdot\dots\cdot\varphi(1)}_{n} \\ &= \underbrace{a\cdot a\cdot\dots\cdot a}_{n} \\ &= a^n. \end{aligned}$$
Therefore, $g\in\langle a\rangle$ $\forall g\in G$, so $G=\langle a\rangle$, & G is cyclic.