Exercises 26: 20, 22, 25, 26, 30, 33, 34

Additional exercises:

1. Let $\phi\colon R \to S$ be a ring homomorphism. Prove that $\ker \phi$ is an ideal of R.

2. The goal of this problem is to prove that if R and S are rings with unity and K is an ideal of $R \times S$, then there are ideals $I \leq R$ and $J \leq S$ such that $K = I \times J$.

   (a) Define $I = \{r \in R \mid \text{ there exists } s \in S \text{ such that } (r, s) \in K\}$ and $J = \{s \in S \mid \text{ there exists } r \in R \text{ such that } (r, s) \in K\}$. Prove that I and J are ideals of R and S, respectively.

   (b) Prove that $K \subseteq I \times J$.

   (c) Prove that $I \times J \subseteq K$. For this part, we want to show that if $r \in I$ and $s \in J$, then $(r, s) \in K$. Consider the elements $s' \in S$ and $r' \in R$ such that $(r, s'), (r', s) \in K$. (Why do such elements exist?) Since K is an ideal, $(a, b)(r, s') \in K$ and $(a, b)(r', s) \in K$ for all $(a, b) \in R \times S$. Find an appropriate choice (or choices) of (a, b) so that you can conclude that $(r, s) \in K$.

3. Let R and S be rings with unity. Show that M is a maximal ideal in $R \times S$ if and only if $M = M_1 \times S$, where M_1 is a maximal ideal in R or $M = R \times M_2$, where M_2 is a maximal ideal in S. You may use that fact that if $I \leq R$ and $J \leq S$ are ideals, then $(R \times S)/(I \times J)$ is isomorphic (as a ring) to $(R/I) \times (S/J)$.

4. Prove that every irreducible polynomial in $\mathbb{R}[x]$ must be degree 1 or 2 using the following steps.

   (a) Define a function $\phi\colon \mathbb{C} \to \mathbb{C}$ by $\phi(a+bi) = a-bi$ (this is the usual complex conjugation). Show that ϕ is a ring homomorphism.

   (b) Suppose $f \in \mathbb{R}[x]$ and suppose $\alpha \in \mathbb{C}$ satisfies $f(\alpha) = 0$. Show that $f(\phi(\alpha)) = 0$, using the fact that ϕ is a ring homomorphism.

   (c) It is known that any $f(x) \in \mathbb{C}[x]$ with degree greater than 0 can be factored as a product of degree 1 polynomials in $\mathbb{C}[x]$. Use this fact and your work above to show that if $f(x) \in \mathbb{R}[x]$ is irreducible, then $f(x)$ has degree 1 or 2. (Hint: suppose $\deg(f) > 1$, and consider $\alpha \in \mathbb{C}$ such that $f(\alpha) = 0$. Apply (b) to conclude that $\phi(\alpha)$ is also a root of $f(x)$. Let $g(x) = (x-\alpha)(x-\phi(\alpha))$ and show that $g(x)$ divides $f(x)$ in $\mathbb{R}[x]$ by using the Division Algorithm in $\mathbb{R}[x]$ – if there is a remainder term $r(x)$, what is $r(\alpha)$? Conclude that $f(x)$ is either reducible or a constant multiple of $g(x)$.)

5. Prove that $\mathbb{R}[x]/(x^2+1) \simeq \mathbb{C}$. There is a hint[1] at the bottom of the page if you don't know how to begin! It would be a good idea to think about the problem before you look at the hint.

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[1]Use the First Isomorphism Theorem for Rings and the evaluation homomorphism $\phi\colon \mathbb{R}[x] \to \mathbb{C}$ defined by $\phi(p(x)) = p(i)$.