Practice Problem Solutions (10/27)

① $f(x,y) = xy$ on $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

The ellipse is a level set of $g(x,y) = \frac{x^2}{8} + \frac{y^2}{2}$.

f will have a max/min on this level set when $\vec{\nabla} f = \lambda \vec{\nabla} g$.

So we have the system of equations:

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ \frac{x^2}{8} + \frac{y^2}{2} = 1 \end{cases} \quad \text{or,} \quad \begin{cases} \langle y, x \rangle = \lambda \langle \frac{x}{4}, y \rangle \\ \frac{x^2}{8} + \frac{y^2}{2} = 1 \end{cases} \quad \text{or,} \quad \begin{cases} y = \lambda \cdot \frac{x}{4} \\ x = \lambda y \\ \frac{x^2}{8} + \frac{y^2}{2} = 1 \end{cases}$$

Solving the 2nd eqn for λ: $\lambda = \frac{x}{y}$. plug into 1st eqn:

$$y = \left(\frac{x}{y}\right)\cdot\left(\frac{x}{4}\right)$$

$y^2 = \frac{x^2}{4}$. Plug into 3rd eqn:

$$\frac{x^2}{8} + \frac{1}{2}\left(\frac{x^2}{4}\right) = 1$$

$$\frac{x^2}{8} + \frac{x^2}{8} = 1$$

$$2x^2 = 8$$

$$x^2 = 4$$

$$x = \pm 2.$$

Since $y^2 = \frac{x^2}{4}$, we plug each value of x in to find y:

$x = +2$: $y^2 = 4/4$
$y^2 = 1$
$y = \pm 1$
$(2,1), (2,-1)$

$x = -2$: $y^2 = 4/4$
$y = \pm 1$
$(-2,1), (-2,-1)$

Find $f(x,y)$ at each pt:

max $(2,1,2)$	min $(-2,1,-2)$
$(2,-1,-2)$ min	$(-2,-1,2)$ max

② $f(x,y) = 3x+4y$ on $x^2+y^2=1$.

$x^2+y^2=1$ is a level set of $g(x,y)=x^2+y^2$.

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$\langle 3,4\rangle = \lambda\langle 2x, 2y\rangle$

$$\begin{cases} 3 = 2\lambda x \\ 4 = 2\lambda y \\ x^2+y^2=1 \end{cases}$$

$3 = 2\lambda x \longrightarrow \lambda = \frac{3}{2x}$ [Note: $x \neq 0$, else $3 = 2\lambda(0) = 0$ & $3 \neq 0$]

Plugging into 2nd eqn yields:

$4 = 2\left(\frac{3}{2x}\right)y = \frac{3y}{x}$

$y = \frac{4x}{3}$.

Plugging into third eqn:

$x^2 + \left(\frac{4x}{3}\right)^2 = 1$

$x^2 + \frac{16x^2}{9} = 1$

$\frac{25}{9}x^2 = 1$

$x^2 = \frac{9}{25}$

$x = \pm\frac{3}{5}$

Using $y = \frac{4x}{3}$:

$x = \frac{3}{5}$: $y = \frac{4}{3}\left(\frac{3}{5}\right) = \frac{4}{5}$

$\left(\frac{3}{5}, \frac{4}{5}\right)$

$x = -\frac{3}{5}$: $y = \frac{4}{3}\left(-\frac{3}{5}\right) = -\frac{4}{5}$

$(-3/5, -4/5)$

Find $f(x,y)$ at these pts:

$\left(\frac{3}{5}, \frac{4}{5}, 5\right)$ max.

$\left(-\frac{3}{5}, -\frac{4}{5}, -5\right)$ min.

③ $f(x,y) = e^{-\frac{1}{3}x^3+x-y^2}$ on $x^2+y^2=4$

$\vec{\nabla} f = \langle -x^2+1, -2y \rangle \cdot e^{-\frac{1}{3}x^3+x-y^2}$

$x^2+y^2=4$ is a level set of $g(x,y)=x^2+y^2$

$\vec{\nabla} g = \langle 2x, 2y \rangle$, and $\vec{\nabla} f = \lambda \vec{\nabla} g$, so

$$\begin{cases} (-x^2+1)e^{-\frac{1}{3}x^3+x-y^2} = \lambda \cdot 2x \\ -2ye^{-\frac{1}{3}x^3+x-y^2} = 2y \cdot \lambda \longrightarrow \lambda = -e^{-\frac{1}{3}x^3+x-y^2} \text{ or } y=0 \\ x^2+y^2=4 \end{cases}$$

If $y=0$, $x^2+0^2=4$

$x^2=4$

$x=\pm 2$

$(2,0), (-2,0)$

If $\lambda = -e^{-\frac{1}{3}x^3+x-y^2}$, plug into 1st eqn:

$(-x^2+1)\cancel{e^{-1/3x^3+x-y^2}} = -2x\cancel{e^{-1/3x^3+x-y^2}}$

[Note: $e^{-1/3x^3+x-y^2} \neq 0$, so ok to divide by it without adding a separate case.]

$-x^2+1 = -2x$

$x^2-2x-1=0$

$x = \frac{2 \pm \sqrt{4-4(1)(-1)}}{2} = \frac{2\pm\sqrt{8}}{2} = \frac{2\pm 2\sqrt{2}}{2}$

$= 1 \pm \sqrt{2}$

Find y:

- $(1+\sqrt{2})^2 + y^2 = 4$

  $1+2\sqrt{2}+2+y^2=4$

  $y^2 = 1-2\sqrt{2}$

  $y = \pm\sqrt{1-2\sqrt{2}}$

  $(1+\sqrt{2}, \sqrt{1-2\sqrt{2}})$
  $(1+\sqrt{2}, -\sqrt{1-2\sqrt{2}})$

- $(1-\sqrt{2})^2 + y^2 = 4$

  $1-2\sqrt{2}+2+y^2=4$

  $y^2 = 1+2\sqrt{2}$

  $y = \pm\sqrt{1+2\sqrt{2}}$

  $(1-\sqrt{2}, \sqrt{1+2\sqrt{2}})$
  $(1-\sqrt{2}, -\sqrt{1+2\sqrt{2}})$

③ (ctd)

Since $1-2\sqrt{2}<0$, $\sqrt{1-2\sqrt{2}}$ is imaginary, so we can ignore these pts.

Find $f(x,y)$ for the remaining 4 pts

$\boxed{(2,0,e^{-2/3})\ \text{min}}$

$(-2,0,e^{2/3})$

$$\boxed{\begin{array}{ll}(1-\sqrt{2},\sqrt{1-2\sqrt{2}},e^{-\frac{7}{3}+\frac{8}{3}\sqrt{2}}) & \text{max}\\ (1-\sqrt{2},-\sqrt{1-2\sqrt{2}},e^{-7/3+8/3\sqrt{2}}) & \text{max}\end{array}}$$